AN ESSAY

ON

THE EFFECTS PRODUCED BY CAUSING WEIGHTS TO TRAVEL OVER ELASTIC BARS.

BY

THE REV. ROBERT WILLIS, M. A., F. R. S., &c.
JACKSONIAN PROFESSOR IN THE UNIVERSITY OF CAMBRIDGE.

(*Extracted from the Appendix to the Report of the Commissioners appointed to inquire into the Application of Iron to Railway Structures.* —*July* 26, 1849.)

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CHAPTER I.

General Remarks and Description of the Apparatus erected in Portsmouth Dockyard, and of the Experiments performed with it by Captain James and Lieutenant Galton.

ONE of the objects to which the attention of the Commission was directed by the terms of its appointment, was "to illustrate by theory and experiment the action which takes place under varying circumstances in iron railway bridges." Now a bridge has necessarily to sustain the action of loads which pass over it, and, in the case of railway bridges, the velocity of transit is exceedingly great.

The effects of loading elastic bars with weights appended to them at rest have been very fully investigated, both by theory and experiment, as is perfectly well known; but the effects produced upon such bars by causing the weights with which they are loaded to travel with more or less velocity along them had never been, as far as the Commissioners were aware, made the subject of research either practically or theoretically. It was therefore resolved that experiments should be arranged for the purpose of determining the influence of velocity communicated to a load upon the deflection and fracture of the structure over which it is transmitted, and which has, therefore, to sustain its pressure during its transit.

It was thought desirable, at the beginning of the investigation, that the experiments should be made on a large scale, so as to give a practical value to the results, whatever they might be, that should be obtained. The object in view was to subject bars of cast iron to the action of passing loads for the purpose of examining how the velocity of any given load would operate to increase or to diminish its pressure upon the bars, and consequently of determining its power in deflecting or fracturing them as compared with the effects of the same load, placed at rest upon the bars in the usual manner of experiments upon the strength of materials.

An apparatus was therefore required which admitted of having bars which were to be the subjects of the experiments readily fixed to receive the passing load, the latter being capable of adjustment to various weights at pleasure; and it was also requisite to have the means of giving any desired velocity to this load. Lastly, contrivances were required for the purpose of registering the effects.

A liberal permission had been granted to us by the Lords Commissioners of the Admiralty to make use of Portsmouth Dockyard for our experiments, and as the apparatus in question required considerable space, it was determined to erect it in that place. Captain H. James, one of Her Majesty's Commissioners for carrying out the present inquiry, also resided at Portsmouth, holding the office of Director of Works in the Dockyard. He, therefore, was requested to undertake the construction of the apparatus required for the purposes already mentioned, and the mechanism about to be described was wholly contrived and set up under his direction. Of this mechanism it is sufficient to say that from the beginning it answered its purpose most admirably, requiring only a few alterations, the necessity for which became evident after the preliminary experiments had shown more clearly the points of the investigation that required to be developed. The experiments themselves were wholly carried out under the personal superintendence of Captain James and Lieutenant Galton, the Secretary to the Commission.

The apparatus was principally designed to experiment on bars of nine feet in length, and the load consisted of a small ordinary railway car, adapted to run on rails three feet asunder, and to receive pigs of cast iron, by which the weight of the whole could be adjusted from half a ton to two tons at pleasure. It was determined to employ an inclined plane as the simplest mode of giving a manageable velocity to the load, and the space at command in the Dockyard enabled this plane to be erected upon a scale that raised its upper extremity 40 feet above the lower part.

The entire machine, together with details of every portion of it, is shown in Plates I. and II. The form and proportions of the car and its rails are sufficiently shown by its side elevation, plan and section, in figs. 4, 5, and 6, respectively. The general form and arrangements of the scaffold are given in figs. 1, 2, and 3.

Figs. 1, 2. This inclined plane or scaffold supported the railroad, of which thirty feet of the upper part were straight and inclined to the horizon at an angle of 46°. The course of the bars was then bent into an arc of a circle of 50 feet radius, by which the upper and inclined part (*A*) of the railroad was gently and imperceptibly connected with (*D D′*) the horizontal portion beneath, which from the point of its junction with the curves was extended 18 feet to the place (*C*) where the ends of the trial bars were fixed. These were laid horizontally so as to form a continuation of the railway, with this difference, that whereas the railway bars were supported by chairs of the ordinary kind, fixed at intervals of 4′ 6″ to the framework of the scaffold, the trial bars were sustained by chairs of a peculiar construction (*F F*) at each end only.

One of these chairs is represented in plan and section on a larger scale in figs. 10, 11, and 12; from which it appears that the end of each trial bar (*C*) was cast with a projection beneath, and kept in its place laterally by a pair of wedges, which were not driven sufficiently tight to impede its vertical deflections. The lower surface of the above-mentioned projection, which formed the bearing surface, could be readily adjusted by the file so as to insure continuity between the upper edges of the fixed rail and of the trial bar respectively at their junction, and thus to avoid the jumping or jerking of the wheels of the car: for it is of the utmost importance to the accuracy of experiments of this kind that the car should enter upon the trial bar without jolting. A wooden wedge was also dropped between the extremities of the rail and trial bar for a similar purpose.

Beyond the farthest end of the trial bars a portion of a similar railway was laid (as will be presently described), for the purpose of receiving the car after it had passed over the bars. Thus the bars formed a part of the railway for the time being, and to determine the effect of any required load and velocity upon the bars, it was only necessary to load the car accordingly, and draw it up to such an altitude of the plane as would correspond to the desired velocity, and, lastly, to release it suddenly. It then ran down the plane and passed over the bars with the velocity acquired, deflecting or fracturing them, as the case might be. From the nature of this apparatus it is necessary to fix a *pair* of trial bars into the frame, for as the car in its passage deflects the bars, it necessarily sinks downwards. If only one trial bar were employed, and the corresponding opposite one stiffened by resting on a sleeper or otherwise, the car would be thrown laterally over. Some inconvenience arises from this necessity for employing two flexible bars at once; but a greater one was occasioned by the fracture of the bars whenever that took place, which of course it frequently did, since one object of the research was to discover the load that would fracture the bars with given velocities. But whenever either bar broke, the car, having lost its support, rolled head over heels into the yard, and usually some hours were consumed in repairing the consequent mischief; also, the fear of such accidents made it necessary for the observers to escape to a safe distance before the car was released, instead of closely watching the phenomena of its passage.

In estimating the load upon the trial bars, it must be remembered that the weight of the car was equally divided between the two, and therefore, although the car was capable of being loaded to two tons, each trial bar could only be exposed to the action of half those weights.

The vertical height of the top of the railway has been said to be 40 feet above the horizontal portion; but the centre of gravity of the car could not, of course, be raised to the very top; and deducting also the retarding effect of friction, it was found that the greatest actual velocity with which the car could be made to pass the trial bars was not greater than 43 feet per second (or about 30 miles per hour), a velocity due to a fall from only 30 feet when resistances are neglected.

The actual velocity of the car was measured by Lieutenant Galton in the following manner:—A distance of 12 feet 6 inches was marked out on each side of the centre of the trial bar (see

Plate II., figs. 4, 5, and 6), on entering which a roller *P*, attached to the car, struck a lever *M*, which, by means of the link rods *M' M'*, pushed the plate *K* from under the pencil *L*, and allowed the latter to come in contact with and trace a line upon the cylinder *O*, which was maintained in equable rotation by an equatorial clock. The arrangement of the pencil, cylinder, and guard-plate *K*, is shown at large in fig. 9. The clock was kindly lent by Dr. Lee, F. R. S., of Hartwell House. When the car had passed to the end of the assigned distance, the roller *P*, striking the lever *N*, raised the pencil by means of the connecting link rod *N'*, the end of which was jointed to an arm hanging from the axis to which the pencil carriage was fixed.

We must now consider the mode of checking the velocity of the car and bringing it to rest, after it had passed over the trial bars. For this purpose the railway was continued beyond the trial bars, exactly in the same manner as in front of them, namely, by a curve and an inclined plane, which is represented in fig. 1, from *D'* to *B*. In the earlier experiments, the car, after passing the trial bars, ran up the second inclined plane, nearly as high as the point whence it had been released from the first. Then it ran down again, again passed over the trial bars and up the first plane, and so backwards and forwards until its velocity became so far subdued that it could be stopped by hand.

But these repeated journeys, besides wasting time, were found to interfere so seriously with the registering apparatus and the adjustment of the trial bars, that a better scheme was carried out at the suggestion of Lieutenant Galton, which is represented in figs. 4 and 5.

A second railway was laid parallel to the first on the horizontal portion, having its bars respectively about nine inches distant from those of the first, and upon the same level. This railway, about 50 feet in length, was curved horizontally to meet the first at its two extremities, and connected to them by switches; the levers and connecting rods of which are shown at *D D*, *D' D'*, figs. 4 and 5. In the position of the apparatus represented in the plan, the switches are set in a position which does not disturb the continuity of the direct line of the rails. If the switches at each end are shifted to the position shown by the dotted lines, the horizontal portion of the direct line which contains the trial bars will be completely cut off, and the railway, descending the inclined plane and curves from each side, will be conducted by the switches to the intermediate railway. (It is plain that the two sets of switches must be shifted.) The mode of performing an experiment with this improvement was as follows :—The switches were, in the first instance, set in the position of the figure, so as to continue the original direct line of rails, and the car, when released, ran down the left-hand inclined plane, and having passed over the trial bars, ran up the second plane to the right. Immediately the two switch levers were shifted so as to cut off the trial bars, and the car, returning, was thus diverted upon the intermediate line upon which it travelled backwards and forwards, running up and down the two curves and inclines as before, but without repassing the trial bars or deranging the registering apparatus.

It remains to describe the apparatus represented in the figures 4, 5, 7, 8, by which the effects and results of the experiments were registered. In the earlier trials it was only thought necessary to ascertain the central deflection of the trial bar, in order to compare its amount as produced *statically* by placing the loaded car at rest upon it, with its amount when obtained *dynamically* by running the same loaded car over it. This deflection was simply obtained by a horizontal lever set at right angles to the middle of the trial bar, and having one end in contact with its lower surface. The other end of the lever carried a pencil, which, when the bar was depressed either statically or dynamically, traced a line upon a piece of paper, the length of which line was proportional to the deflection.

But upon investigating the theory of these experiments I soon perceived that the information thus conveyed was wholly inadequate, and that much more information was required of the movements imparted to the bar by the passing weight. At my suggestion, therefore, the registering apparatus represented in the figures was substituted for the simple deflectograph above described. The reasoning which led me to the contrivance of this apparatus will be fully explained below (see p. 457), and although, as it will appear, its construction was not sufficiently delicate to carry out my purposes as originally intended, it was employed for the whole of the subsequent series of experiments. Five pencils were attached to as many points of the trial bar, equidistant from each other and from the ends of the bar. In the section, fig. 7, *C* is the trial bar, and a spring pencil appears beneath it, the tube of which is fixed to a clamp that can be readily screwed to the lower part of the bar so as not to be displaced by the flanges of the car-wheels in their passage along the upper surface of the bar.

A long board, *E E*, is placed in front of and parallel to the bar at a distance of about an inch and a quarter. This board, six inches in width (or rather height), is arranged as shown by the section, so as to run easily upon rollers in the direction of its length. Its inner vertical surface (or that which lies next to the bar) is covered with paper and receives the traces of the five pencils; for, as shown in fig. 5, the board *E* is sufficiently long to be in contact with all the pencils *a, b, c, d, e,* at the same time. If the board remained at rest during the passage of the car, it is plain that each pencil would trace a line upon the paper, which would be equal to the deflection of the corresponding point of the bar to which that pencil was fixed; and thus, instead of recording merely the central deflection of the bar, the apparatus would inform us of the deflection of each of the five points of the bar. Let us now suppose that a slow equable motion is given to the board, which, as already explained, is mounted on rollers. In this case each pencil will, in lieu of a simple vertical line, trace a curve in the form of a loop or irregular U, the inflection of which will, when properly analyzed, inform us of every particular respecting the motion of the bar, as I shall explain at length below.

However, to do this completely, the board must be maintained in motion with a constant velocity such as an equatorial clock or similar contrivance alone can effect, and that only when the board and its rollers are so mounted as to move with small and equable friction, a condition which the general roughness of the apparatus in question rendered inadmissible. The board, therefore, was simply fitted to receive its motion from the descent of a weight at *G*, (figs. 4, 5, 8,) fastened to a string, which, passing over three pulleys, was thereby conducted into the proper horizontal direction, and also to the level of the board, to the end of which it was tied. The weight was temporarily prevented from descending by a small board placed under it, and which was connected to a lever *H*, as shown in the figures, in such a manner that when the car in its course arrived at this lever, near to the trial bar, it struck it aside, and thus drawing the board from beneath the weight, the latter began its descent, dragging with it the board. The board thus received a travelling motion, of course considerably accelerated, but which enabled it to receive from the pencils curves of the nature of those above described.

These curves, although from the irregular motion of the board they were inadequate to convey the entire information for which I sought, did yet suffice to record the simultaneous deflections of each of the five points, and were used for this purpose alone. The remaining information I contrived to obtain by means of my own, which will be presently described.

Upwards of four hundred* experiments were made with this apparatus by Captain James and Lieutenant Galton, and the results which they obtained were equally new and important, developing, for the first time, the fact that a given weight passing rapidly along a bar produces a greater deflection in that bar during its passage, than it would have done had it been suspended at rest from the centre of the bar.

The three first series of experiments were made upon bars of Blaenavon cast iron, nine feet long, of which those of the first series were an inch broad and two inches deep. In the second they were one inch broad by three inches deep, and in the third four inches broad by an inch and a half deep. As these three series were each managed in the same manner, it will only be necessary to describe one at length, for which purpose I shall select the second series.† In describing the load of the car, it must be remembered that its actual weight is distributed upon four wheels, two of which rest on each trial bar. Thus, when the weight of the car and its load amount to 2240 lbs., each bar is loaded with 1120 lbs. In describing the experiments, therefore, the weight mentioned must be understood to be the total weight of the loaded car.

In the first place, a sufficient number of bars having been cast

* In this enumeration each journey of the carriage is reckoned as one experiment. But in the Tables the experiments are arranged in groups of seven or eight of such journeys, each group being numbered as one experiment, so that the total number of experiments appears much less.

† See the Tabular Summary below, p. 445, Second Series, Experiment No. 7.

of the above-stated dimensions, a pair of them were placed in the chairs, and the car having been set at rest upon their centre, was loaded with gradually increasing weights from 1120 lbs. upwards, until one of the bars broke, the deflections and sets having been carefully noted for each accession of weight. This preliminary experiment, which was repeated upon three pairs of bars, was made for the purpose of testing in the usual manner the actual strength of the bars which were to be the subject of the dynamical experiments. It was thought better to test in this manner the quality of specimens taken at random from the actual parcel of bars provided for the dynamical experiments than to trust to calculated results.

A pair of bars were, in the next place, selected for a dynamical trial, and placed in the chairs. The car was loaded with 1120 lbs. and placed at rest in the centre of the bars. The statical deflection was 0·32 inch. The car was then drawn up to the point of the inclined plane which corresponded to a velocity of 29 feet per second, or 20 miles per hour, and suddenly released. The transit over the bars produced a deflection of 0·36 inch. The velocity given to the load had thus added one-tenth to the statical deflection. The car was then loaded to 1778 lbs., 2348 lbs., 2670 lbs., and so on, adding 56 lbs. each time, and always releasing it from the same point of the plane, the deflection meanwhile steadily increasing at each increase of weight, until, with a load of 2999 lbs., it became 2·67 inches. This load would, calculating from the statical deflection of the same bar by 1120 lbs., have produced a statical deflection of 1·30 inch. The velocity, therefore, in this case more than doubled the statical deflection due to the load. The car, as already observed, was always drawn up to the same point, so that the velocity remained constant in each set of experiments with a given pair of bars, and the load was increased at each successive trial until one or both bars broke. In the set of experiments we are now considering, the next load of 3167 lbs. fractured both bars at once. The mean statical breaking weight of bars of these dimensions is about 4200 lbs. Thus it is shown that the motion of the load over the bars increases the deflection, and, as would naturally follow, enables a smaller weight to fracture them. When higher velocities are given to the car, the above effects are increased.

A pair of bars received from a load of 1120 lbs. a statical deflection of 0·27 inch.* When a velocity of 43 feet per second (30 miles per hour) was given to the car, the deflection became 0·52 inch; and with loads of 1778 lbs. and 2066 lbs., it reached 1·07 inch and 1·87 inch respectively. The bars were fractured with 2122 lbs.; their mean statical breaking weight being about 4200 lbs. Calculating the statical deflection due to the above loads, it appears that this high velocity enabled 1778 lbs. to effect more than double that deflection, and 2066 lbs. to increase it threefold.

To estimate the increase of the statical deflection, produced by the velocity of the load in the above examples, it is necessary, as I have shown above, to know the statical deflection due to each load. Now the object of the experimenters was simply to ascertain the breaking weight of each pair of bars under a given velocity. They, therefore, only tried the statical deflection of each pair with the first load of 1120 lbs.; for in dealing with cast iron, the imperfection of its elasticity and the consequent amount and irregularity of the set makes it necessary to avoid as much as possible the repeated deflections of the bars. On this account they did not ascertain the statical deflection for each successive load, but contented themselves, after the first trial, with releasing the car from its constant altitude, increasing the load at each trip until the bars broke. The statical deflection, therefore, after the first, can only be calculated by comparing that first deflection due to 1120 lbs. with the deflections in the preliminary statical experiments, already described, which were made for this purpose.

The irregularities introduced by the set of the bars, which our imperfect knowledge of that phenomenon makes it impossible for us to remove from the calculations, must prevent this method from being very accurate, but it will be found sufficiently exact to enable us to compare roughly the statical with the dynamical deflection, considering the other sources of irregularity and error which are inseparable from experiments of this nature, as I shall point out below.

If indeed the whole of the bars could be cast of the same strength, the deflection of one bar would correspond so nearly to those of the other that no sensible error need be apprehended,

* See Second Series, p. 446, Experiment 14.

but this can never be the case. Compare, for example, the three experiments in the second series upon a velocity of 15 feet with the three following upon a velocity of 29 feet, and it will appear that the statical deflections due to 1120 lbs. in these six experiments vary from ·29 to ·42, although all the bars were cast in the same mould.

But to compare the effects of velocity upon the deflections with more accuracy, some experiments were subsequently undertaken upon a different principle, namely, that in each set the velocity should be varied, and the load remain constant; thus the statical deflection due to this constant load, being ascertained at the beginning, was applicable without error to the whole: these are contained in the sixth and seventh series of experiments.*

Within the limits employed in the previous experiments, the increase of velocity had been constantly accompanied by an increase of deflection, but it was conceivable that with a very high velocity the load might pass over the bar without having time even to fall through the space required for the statical deflection, and that thus there must be a limit to the increase of the deflection, so that beyond the velocity corresponding to this limit, the deflection would diminish. It was clear that this limit would be approached more nearly by employing shorter bars, and those as flexible as possible, for the purpose of at once diminishing the time of passage, and increasing the space through which the load must fall vertically. Bars of wrought iron were tried, 4 ft. 6 ins. in length; and in order to get rid of the complication of effect produced by having two wheels pressing on the bar at once, the car was elongated, so as to render the distance between its axles 6 ft. 6 ins.

The load therefore still pressed upon each rail with two wheels, but as the trial bar was shorter than the distance between these wheels, the travelling load could only press upon it in one point at a time, for the front wheel had completely passed off the bar before the hind wheel entered upon it. The load was laid so as to press much more upon the front than upon the hind wheel, and thus the effect of the passage of the latter was insignificant. The desired maximum deflection was not, however, reached by these bars, as will be seen by referring to the Tables in the Report (pp. 239, 240). But a pair of steel bars 2 ft. 3 ins. long, 2 ins.

* See Table X. below, p. 488.

broad, and $\frac{1}{4}$ in. deep, gave the following results, and exhibited the effects which were sought for:

Velocity, in feet, per second . ..		15	24	29	34	44
Central Deflection	·70	1·02	1·32	1·45	1·30	1·03

A bar of wrought iron 9 ft. long, 1 in. broad, and 3 ins. deep, with a load of 1778 lbs., gave the following relations between the velocities and deflections, in which the latter pass the maximum limit:

Velocity, in feet, per second		15	29	36	43
Central Deflection	·29	·38	·50	·62	·46

The following Tables contain a Summary of the central deflection in the three first Series of the Portsmouth Experiments, showing the velocities and weights employed, the statical deflections due to those weights, the dynamical deflections obtained, and the ratio between the statical and dynamical deflections in each case.

The bars were all 9 feet long between the supports: the first column in the following Tables contains the number corresponding to each experiment (or rather set of experiments) in the detailed Tables given in the Appendix to the Report, p. 215. The second column gives the weight upon each pair of bars. The third column contains the statical central deflection due to the weight. The first deflection in each experiment which corresponds to the weight of 1120 lbs. was obtained by trial, the remainder for the higher weights, calculated as explained above. The fourth column contains the dynamical deflections given by the experiment. Finally, the fifth column is the ratio of the dynamical to the statical deflection.

Each experiment was terminated necessarily by one or both bars breaking. This fact is recorded by the word "*broke*," inserted in that part of the Table which belongs to the fractured bar.

TABLE I.—FIRST SERIES.

Bars 1 inch broad, 2 inches deep.

No. of Experiment.	Weight in lbs.	Left Bar.			Right Bar.		
		Statical deflection.	Dynamical deflection.	Ratio.	Statical deflection.	Dynamical deflection.	Ratio.
		Velocity 15 feet per second.					
4	1120	·88	1·24	1·41			
	1240	1·10	1·70	1·54			
	1440	1·48	1·98	1·34			
	1560	1·71	2·51	1·47			
	1760	2·09	3·00	1·43			
	1876	Broke.					
5	1120	·86	1·11	1·28			
	1240	1·06	1·41	1·33			
	1440	1·45	1·94	1·24			
	1560	1·66	2·50	1·55			
	1760	2·03	3·06	1·51			
	1788	2·10	3·53	1·68			
	1816	2·12	3·61	1·70			
	1844	2·18	4·17	1·91	Broke.		
6	1120	·62	·74	1·19			
	1240	·77	·88	1·14			
	1356	·93	1·10	1·18			
	1460	1·07	1·34	1·25			
	1560	1·20	1·76	1·47			
	1680	1·36	2·37	1·74			
	1792	1·51	2·90	1·92			
	1816	Broke.					
		Velocity 24 feet per second.					
7	1120	·64	1·02	1·59			
	1240	·80	1·55	1·94			
	1356	·96	2·70	2·81			
	1412	1·04	3·16	3·04			
	1440	Broke.	..	..	Broke.		
8	1120	·65	·87	1·43	·88	1·10	1·25
	1240	·81	1·10	1·35	1·10	1·30	1·18
	1356	·98	2·32	2·37	1·32	1·60	1·21
	1412	1·06	2·85	2·69	1·43	2·43	1·70
	1440	1·09	3·81	3·50	1·48	2·76	1·86
	1468	1·13	..	..	1·54	2·88	1·87
	1496	1·17	3·94	3·37	1·59	2·94	1·85
	1524	Broke.	..	..	Broke.		

FIRST SERIES—(*continued*).

Bars 1 inch broad, 2 inches deep.

No. of Experiment.	Weight in lbs.	Left Bar. Statical deflection.	Left Bar. Dynamical deflection.	Left Bar. Ratio.	Right Bar. Statical deflection.	Right Bar. Dynamical deflection.	Right Bar. Ratio.
Velocity 24 feet per second.							
9	1120	·74	1·14	1·54	·72	1·10	1·52
	1240	·92	1·47	1·59	·90	1·30	1·44
	1356	1·10	1·74	1·58	1·08	1·68	1·55
	1412	1·20	2·02	1·70	1·17	1·70	1·45
	1440	1·24	2·23	1·80	1·21	2·00	1·65
	1468	1·29	2·41	1·87	1·25	2·36	1·89
	1496	1·33	2·54	1·90	1·29	2·74	2·12
	1520	1·37	2·68	1·96	1·33	3·00	2·26
	1552	1·42	2·77	1·95	1·38	3·24	2·35
	1580	1·46	3·08	2·11	1·42	3·60	2·54
	1604	Broke.	..	..	Broke.		
Velocity 29 feet per second.							
10	1120	·95	1·80	1·89	1·80	2·10	2·10
	1240	Broke.	..	..	Broke.		
11	1120	1·17	2·54	2·17	·75	2·04	2·71
	1176	1·31	3·36	2·56	·84	2·65	3·15
	1204	Broke.	..	..	·89	3·10	2·76
12	1120	·96	2·30	2·39	1·18	2·04	1·72
	1176	1·08	3·03	2·80	1·32	2·68	2·03
	1204	Broke.	..	..	Broke.		
Velocity 33 feet per second.							
13	1120	·84	2·02	2·40	·73	1·86	2·55
	1176	·94	2·67	2·83	·82	2·26	2·76
	1204	Broke.	..	..	·87	2·60	2·99
14	1120	·81	1·31	1·61	·70	1·15	1·64
	1176	·91	1·86	2·05	·78	1·50	1·92
	1204	·96	2·44	2·54	·83	1·91	2·28
	1232	1·00	3·02	3·02	·87	2·46	2·83
	1260	1·04	3·65	3·51	·90	2·80	3·11
	1288	Broke.	..	..	·95	2·90	3·05
15	1120	1·30	3·04	2·34	1·12	2·48	2·21
	1148	Broke.					
Velocity 36 feet per second.							
16	1120	·86	1·86	2·16			
	1148	·94	2·25	2·38	Broke.		

FIRST SERIES—(*continued*).

Bars 1 inch broad, 2 inches deep.

No. of Experiment.	Weight in lbs.	Left Bar. Statical deflection.	Left Bar. Dynamical deflection.	Left Bar. Ratio.	Right Bar. Statical deflection.	Right Bar. Dynamical deflection.	Right Bar. Ratio.
Velocity 36 feet per second.							
17	1120	·72	1·64	2·28	·70	1·50	2·14
	1148	·76	2·26	2·97	·74	2·08	2·80
	1176	Broke.	..	..	Broke.		
18	1120	·70	1·50	2·14	·70	1·40	2·00
	1148	·74	2·10	2·83	·74	1·73	2·33
	1176	·78	2·31	2·76	·78	2·14	2·74
	1204	Broke.	..	..	Broke.		

TABLE II.—SECOND SERIES.

Bars 1 inch broad, 3 inches deep.

No. of Experiment.	Weight in lbs.	Left Bar. Statical deflection.	Left Bar. Dynamical deflection.	Left Bar. Ratio.	Right Bar. Statical deflection.	Right Bar. Dynamical deflection.	Right Bar. Ratio.
Velocity 15 feet per second.							
4	1120	·37	·41	1·1	·39	·41	1·05
	1778	·69	·58	·87	·73	·70	·96
	2348	1·02	·97	·95	1·07	1·00	·93
	2955	1·47	1·65	1·11	1·55	1·46	·94
	3296	1·74	2·35	1·34	1·84	1·95	1·06
	3352	1·80	2·70	1·5	1·90	2·38	1·25
	3408	Broke.					
5	1120	·38	·42	1·10	·44	·47	1·06
	1778	·71	·69	·97	·82	·72	·88
	2348	1·05	1·02	·97	1·21	1·02	·84
	2955	1·51	1·66	1·10	1·74	1·58	·91
	3296	Broke.	..	..	..	1·72	
6	1120	·29	·31	1·07	·27	·36	1·33
	1778	·54	·60	1·11	·51	·60	1·18
	2348	·80	·83	1·04	·75	·80	1·07
	2955	1·19	1·50	1·26	1·07	1·15	1·07
	3296	1·37	1·85	1·35	1·28	1·32	1·03
	3408	1·46	2·22	1·52	1·36	1·45	1·06
	3464	1·50	2·65	1·76	1·40	1·56	1·11
	3496	Broke.	..	..	..	1·82	
Velocity 29 feet per second.							
7	1120	·32	·36	1·11	·32	·42	1·31
	1778	·60	·76	1·26	·60	·86	1·43
	2348	·88	1·36	1·52	·88	1·34	1·52
	2670	1·07	1·82	1·70	1·07	1·78	1·66
	2775	1·14	2·06	1·80	1·14	1·88	1·65

SECOND SERIES—(*continued*).

Bars 1 inch broad, 3 inches deep.

No. of Experiment.	Weight in lbs.	Left Bar.			Right Bar.		
		Statical deflection.	Dynamical deflection.	Ratio.	Statical deflection.	Dynamical deflection.	Ratio.
Velocity 29 feet per second.							
7	2831	1·18	2·16	1·83	1·18	1·91	1·62
	2887	1·22	2·27	1·86			
	2943	1·26	2·52	2·00			
	2999	1·30	2·67	2·05			
	3167	Broke.	..	..	Broke.		
8	1120	·42	·54	1·28	·45	·64	1·42
	1778	·79	1·19	1·50	·14	1·09	1·30
	2348	1·15	2·02	1·75	1·24	1·57	1·26
	2955	Broke.	..	..	Broke.		
9	1120	·33	·52	1·57	·32	·42	1·31
	1778	·62	·88	1·42	·60	·76	1·26
	2348	·91	1·59	1·75	·88	1·58	1·80
	2955	1·31	2·77	2·07	1·27	2·01	1·58
	3011	Broke.	..	..	Broke.		
Velocity 36 feet per second.							
10	1120	·39	·67	1·71			
	1778	·73	1·12	1·53			
	2348	1·07	2·08	1·94			
	2468	Broke.	..	..	Broke.		
11	1120	·34	·50	1·47	·37	·58	1·56
	1778	·62	1·09	1·75	·69	1·03	1·49
	2348	·92	1·90	2·05	1·02	1·78	1·73
	2404	Broke.					
12	1120	·49	·72	1·47	·40	·72	1·8
	1778	·93	1·31	1·42	·75	1·54	2·05
	2348	Broke.	..	..	Broke.		
Velocity 43 feet per second.							
13	1120	·34	·44	1·29	·30	·46	1·53
	1778	·62	·93	1·50	·56	1·18	2·10
	2066	·76	1·56	2·04	·68	1·84	2·67
	2182	Broke.	..	..	Broke.		
14	1120	·27	·52	1·92	·30	·68	2·27
	1778	·51	1·07	2·10	·56	1·30	2·31
	2066	·61	1·87	3·07	·68	2·00	2·94
	2122	Broke.	..	..	Broke.		
15	1120	·24	·38	1·58	·26	·50	1·92
	1776	·45	·86	1·90	·50	1·02	2·04
	2066	·55	1·30	2·35	·59	1·40	2·36
	2182	·60	1·86	3·09	·65	2·02	3·11
	2242	Broke.	..	..	Broke.		

TABLE III.—THIRD SERIES.

Bars 4 inches broad, $1\frac{1}{2}$ inch deep.

No. of Experiment.	Weight in lbs.	Statical deflection.	Dynamical deflection.	Ratio.
Velocity 15 feet per second.				
3	1120	·43	·63	1·46
	1778	·83	1·35	1·64
	2348	1·27	2·00	1·57
	2955	1·88	3·78	2·01
	3191	2·17	4·65	2·14
	3247	2·23	4·85	2·17
	3303	Broke.		
4	1120	·57	·78	1·36
	1778	1·10	1·45	1·32
	2348	1·71	2·21	1·29
	2955	2·54	4·12	1·62
	3296	3·04	4·85	1·59
		Right bar broke.		
Velocity 29 feet per second.				
5	1120	·74	1·08	1·45
	1778	1·44	2·04	1·42
	2066	1·82	2·92	1·43
	2348	2·21	4·14	1·87
	2670	Both bars broke.		
6	1120	·60	1·01	1·68
	1778	1·16	2·17	1·87
	2348	1·80	3·72	2·06
	2670	Broke.		
Velocity 36 feet per second.				
7	1120	·52	·95	1·82
	1778	1·00	2·19	2·19
	2060	1·26	3·88	3·08
	2176	Broke.		
8	1120	·58	1·23	2·11
	1778	1·12	3·11	2·78
	2060	Both bars broke.		
Velocity 43 feet per second.				
9	1120	·63	1·54	2·45
	1778	Both bars broke.		
10	1120	·50	1·28	2·56
	1402	·69	2·31	3·35
	1522	·77	3·18	4·13
	1638	·85	4·39	5·14
		Right bar broke.		

The mode in which the bars were fractured in the above experiments is delineated in Plate III. It will be seen that the fractures took place, with few exceptions, at points beyond the centre of the bar, and that the bars were usually broken into three, and often into four or five pieces, thus indicating a great and violent strain towards the end of the transit of the load, which will be found in perfect accordance with the theoretical and experimental results given in the succeeding chapters. It must be observed that in all the examples of the third series, in which broad thin bars are used, there is but a single fracture, and that always beyond the centre.

The results which we have passed in review were obtained from horizontal straight bars. But it was suggested that if the bars were curved, or made convex upwards, the increase of deflection produced by the velocity of the load would be certainly diminished, and might be entirely removed; for as the effects in question are analogous to the centrifugal action of bodies moving on curves, if the bar were curved into such a form that the weight of the load should depress it exactly to the horizontal line, passing through its bearing points, then this centrifugal action would be completely destroyed. And if this were not exactly effected, the convex curvature would diminish the pressure of the moving load. It was for the purpose of following out these views that the 'Eighth Series of Experiments,' namely, upon curved bars, which will be found in pages 241 to 244 of the Parliamentary Report, were undertaken. They show a considerable reduction in the increment of deflection produced by the velocity of the load, but they were not carried far enough to lead to complete results.

It is very doubtful whether in practice difficulties would not be introduced by the attempt to curve the rails, that would counterbalance the diminution of deflection. A bad joint or sudden change of direction in the rails has a much greater effect in enabling the carriages to shake and strain the bridge than the velocity of the load can possibly produce. Now although the bridge may be, and indeed generally is, curved or cambered upwards in a slight degree, the rails are laid in straight lengths. Thus they form a portion of a polygon with very obtuse angles, and a carriage travelling with the high velocity employed on railways is necessarily at each angle of this polygon, that is, at each joint of the rail, projected onwards in the direction of the rail it has left, so as to fall, in a small parabola, upon the next rail, with a blow that, repeated as it is by the continually passing carriages, gradually serves to deteriorate and disarrange the joints of the railway. This effect is very observable, and in the experiments of the Commission upon Ewell Bridge I was able to detect it by the jumping of the engine, &c. during the passage of the train, while I was stationed beneath the bridge to watch the deflectograph. The rails of this bridge are carefully laid with good joints, but the rails, as above described, are straight, and the bridge cambered.

The experiments upon Ewell and Godstone Bridges, the results of which are given below,* were made for the purpose of comparing the startling and unexpected results obtained at Portsmouth with some cases of real practice in order to discover

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* *Experiments made by the Commissioners on the Ewell and Godstone Bridges.*

The apparatus employed in making these Experiments is detailed in Plate IV.

Ewell Bridge. (*Epsom and Croydon Railway.*)

Span, 48 feet.
Two girders to support each line of rails.
Depth of girders at centre, 3 feet 6 inches.
Width of bottom flange, 20 inches.

	Tons.
Thickness of do., 3 inches.	
Weight of two girders	20
Weight of platform between these girders	10
Total weight of half the bridge	30
Weight of engine	25·2
Weight of tender	13·8
Total	39

Velocity in feet per second.	Deflection in decimals of an inch.
0	·215
25	·215
30·9	·23
32·3	·225
53·7	·245
75	·235

The deflections do not increase steadily, but this could hardly be expected from the many causes of disturbance.

whether an increase of deflection was to be found in actual bridges of the same nature and amount as those which exhibited themselves upon the 9-feet bars. It will be seen that in the Ewell Bridge, the span of which is 48 feet, the statical deflection produced by the engine and tender was only 0·215 in. This was increased to 0·245 in. by a velocity of 54 feet per second, or about 35 miles per hour. A velocity of 75 feet gave a somewhat less deflection, namely 0·235 inch:—

$$\text{Hence} \qquad \frac{\text{greatest dynamical deflection}}{\text{statical deflection}} = 1{\cdot}14,$$

exhibiting an increase of about one-seventh.

In the case of the Godstone Bridge, the span was 30 feet, the statical deflection produced by the engine and tender was 0·19 in., and the dynamical deflection due to a velocity of 73 feet per second was 0·25 inch.

$$\text{Hence} \qquad \frac{\text{dynamical deflection}}{\text{statical deflection}} = 1{\cdot}315,$$

showing an increase of little short of one-third.

In experiments of this kind the deflections must be ascertained

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Godstone Bridge. (South Eastern Railway.)

Span, 30 feet.
Three girders support the roadway.
Depth of girders at centre, 3 feet.
Width of bottom flange, 15 inches.

	Tons.
Thickness of do., $2\frac{1}{4}$ inches.	
Weight of two girders	15
Weight of platform between these girders . . .	10
Total weight of half the bridge . .	25
Weight of engine	21
Weight of tender	12
Total	33

Velocity in feet per second.	Deflection in decimals of an inch.
0	·19
22	·23
40	·22
73	·25

very carefully, for they are so small that the increase may escape notice altogether if roughly measured. Yet it must be remembered that the increase of pressure on the bridge produced by the dynamical action is measured by the increase of the deflections, however small the deflections themselves may be. We therefore selected bridges which were built to carry railways over roads, so that we could erect a temporary scaffold upon the road that should be perfectly independent of the flexure of the bridge above, and of easy access, (Plate IV.) Upon this scaffold was fixed a vertical drawing-board to receive the trace of a pencil, clamped to the lower edges of one of the girders of the bridge. Thus the pencil during the passage of the engine and tender traced a vertical line equal to the deflection. The board was constructed so as to admit of being shifted horizontally after each deflection had been traced, and thus to be ready to receive the trace of the next. The pencil was carefully watched during the passage of the load to guard against accidental jerks or shifts of the apparatus, which, however, were not found to happen.

TABLE OF VELOCITY.

Velocity in feet per second.	Velocity in miles per hour.	Height in feet due to Velocity.
10	6·82	1·55
15	10·2	3·49
20	13·6	6·21
30	20·5	13·97
40	27·3	24·8
44	30·	30·05
50	34·1	38·82
60	40·9	59·00
70	47·7	76·08
80	54·5	99·37
88	60·	120·24
90	61·4	125·77
100	68·2	155·27

The foregoing Table may be useful for reference during the reading of this Essay, to compare velocities, measured in feet and miles respectively.

CHAPTER II.

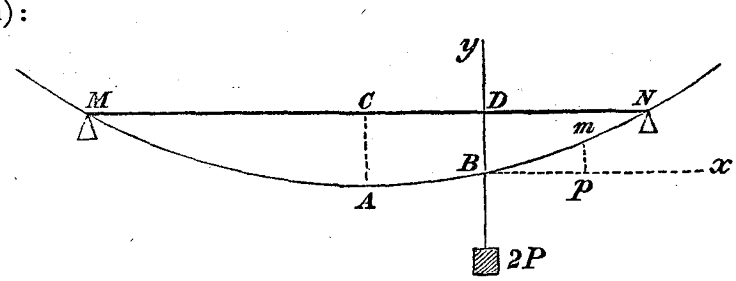
On the general Nature of the Problem, and on the Apparatus employed by me at Cambridge to obtain the Trajectory mechanically.

HAVING now explained the apparatus employed at Portsmouth, and the remarkable results which it has produced, it remains to examine the laws which connect the phenomena, in order to extend them to larger structures, and ascertain the effects of moving loads upon actual bridges. A few simple mechanical considerations will explain the method in which I shall proceed to investigate this part of the subject.

Let A, B, fig. 1, Plate V., be two fixed props at the same horizontal level, upon which an elastic bar, $A\ B$, rests. This bar is of equal section throughout, and its weight is supposed to be so small that it may be neglected. If a weight, W, be suspended to any given point, P, of the bar, it will depress it, and cause the bar to assume the form of a certain curve, $A\ P\ D\ E\ B$, of which the equation is known.* The principal properties of this curve with which we are at present concerned are as follows:—

1. It is convex downwards throughout.
2. The greatest curvature is at the point of suspension of the

* The equation to this curve is given by Navier, 'Application de la Mécanique à l'Etablissement des Constructions et des Machines,' Paris, 1833, tom. i. p. 231, in the following form (with a slight modification of the notation):



$M\ N$ the two props, $M\ A\ B\ N$ the bar loaded with a weight, $2\ P$, which is suspended to a point B, not in the centre.

Let $M\ N = 2\ a$, C the centre of the bar, $C\ D = z$, $B\ p = x$, $m\ p = y$, $B\ D$

weight, P; and this is the point at which the bar would break if the weight were increased sufficiently to produce rupture.

3. If the weight be suspended from the centre, Q, its point of suspension will coincide with the point of the greatest deflection of the bar, and the curve will be symmetrical. But if the point of suspension be out of the centre, as at P in the figure, then it will no longer be the point of the greatest deflection. This greatest deflection, or maximum ordinate of the curve, will be found at M, between the point of suspension, P, and the centre of the curve, D, but much nearer to the latter. In fact, it can be shown that whatever be the horizontal distance of the point of suspension from the centre of the bar, the distance of the point M from the centre can never be greater than 0·154 of the half-length of the bar.

4. A given weight, W, suspended to the bar, will produce a greater or less amount of deflection in the entire bar, according as its point of suspension is nearer to or farther from the centre respectively, and, consequently, the greatest deflection of all when suspended from the centre itself.

5. The deflection of the point of suspension itself can be shown

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(the deflection of the point of suspension below the horizontal line) $= f$, the angle which the tangent to the curve of the bar makes at B with the horizon $= w$, the deflection which the weight $2\,P$ would produce in the bar if suspended from the centre $= S$. Then it can be shown that for the part of the curve $B\,N$ we have

$$y = 3\,S \, . \, \frac{a+z}{a^4} \left\{ \tfrac{2}{3} \overline{a - z} \, . \, z x + \tfrac{1}{2} \, . \, \overline{a - z} \, . \, x^2 - \tfrac{1}{6} \, . \, x^3 \right\};$$

the equation to the other part of the curve, $B\,M$, will be found by writing z negative. We have also

$$f = \frac{S}{a^4} \left(a^2 - z^2\right)^2 \, . \quad \tan. w \, . = \frac{2\,S}{a^4} \left(a^2 - z^2\right) z.$$

The value of x, which corresponds to the greatest deflection of the curve below the horizontal line, is given by the equation

$$x = a + z - \sqrt{a^2 + \tfrac{2}{3} a z - \tfrac{1}{3} z^2},$$

in which x is measured backwards from B towards M. If the point of suspension be gradually shifted nearer to N, this ordinate of greatest deflection will increase its distance from the centre of the curve, which distance will be the greatest when B coincides with N, in which case $z = a$, and we obtain ·154 × a for the distance of the ordinate of greatest deflection from the centre of the curve.

to vary directly as the weight, W, multiplied by the square of the product of the segments into which the point of suspension divides the bar (supposing, which is always the case in the subject under consideration, that the deflection is small compared with the length of the bar). The most convenient expression for the deflection of the point of suspension is the following. Let P be the point of the bar from which the weight, W, is suspended, and $AN = x$, $NP = y$, be its co-ordinates; let a be half the length of the bar, or half the distance between the props; S the deflection which the weight, W, would produce in the centre of the bar, if suspended there: then, because the ordinate y is the deflection of the suspending point P, and this ordinate divides the line AB into the segments x and $2a - x$, we have, from what has been above stated,

$$S : y :: a^4 : x^2 (2a - x)^2 \therefore y = \frac{S}{a^4} (2ax - x^2)^2.$$

Having laid down these principles, which are derived from writers on the strength of materials, let us suppose the point of suspension of the given weight, W, to be shifted in succession to a series of points along the length of the bar, lying pretty close together. If a board covered with paper be fixed behind the bar, so as just to leave space for freedom of motion in the latter, and if these successive points of suspension be marked upon the paper, we shall obtain a dotted line, $APQRB$, as shown in the figure, which is the *locus* of the points of suspension; and of course, if the successive points be taken in sufficient number to lie very close together, we obtain a continuous curve for this *locus*. It is easy to see that the expression obtained above for the amount of deflection produced at the suspending point by a given weight, namely,

$$y = \frac{S}{a^4} (2ax - x^2)^2, \text{ is, in fact, the equation to this } locus.$$

It is better, perhaps, to conceive the weight to be a small heavy cylindrical body resting on the upper flat surface of the bar, and capable of rolling along it, instead of being suspended by a hook, as the former hypothesis approaches nearer to the actual problem which we have to solve, namely, the travelling of a carriage along a bridge. It will thus be perceived that the dotted curve is the path, or *trajectory*, which the centre of this body describes in space during a very slow and gradual passage along the bar, or, rather, a shifting motion from one end to the other, point by point. This form of the trajectory only corresponds to the very slowest continued motion of the body along the bar. Always supposing the body to travel with a uniform motion from one end to the other, the slightest increase of its velocity produces a change in the form of the trajectory, which change is greater as greater velocities are taken. The exact nature and amount of this change under different circumstances will be shown below, as well as the methods by which it was determined, but the general effect is, that the curve is no longer symmetrical to the centre; the greatest depression of this curve being thrown into the second half of it, while the first half is less depressed than with the slow motion. The dotted curve $A P Q R B$, above described, is the form of the trajectory, which is the limit to all these forms, and corresponds to the very slowest motion, or, rather, to the shifting motion of the weight, in which the system is in statical equilibrium at each successive position of the load. On the other hand, the dotted curve $A G H K L$ is one of the forms which the trajectory assumes when velocity is imparted to the body. To distinguish the first form of the trajectory from the others, I shall term it the *equilibrium trajectory*. The object of the investigation which follows is to examine the form and proportion of these trajectories in general, under different relations between the elasticity, dimensions, and weight of the bar, and the magnitude and velocity of the load; first describing the experimental inquiry, and next proceeding to the theoretical principles by which the laws of the phenomena and the *modus operandi* of the forces which are called into action may be developed.

It must be carefully observed that the equilibrium trajectory is a totally different curve from the curve into which the bar is bent at every different position of the weight. In fact, the two curves only coincide at two points, namely, that at which the weight is suspended, and a point at the opposite end. These two points of intersection merge into one, and become a point of contingence at the instant the body passes the centre.* Thus the point at

* It may be useful to mention that from the equation of the equilibrium curve (z), it can be shown easily that its radius of curvature at each extremity

which the equilibrium trajectory *touches* the curve of the bar corresponds to the greatest deflection of the bar.

When we know the form of the trajectory under any of its phases, whether as the equilibrium curve or as the curve corresponding to any given velocity, we can also find the form of the bar at any moment; for the bars are so stiff and the deflections so small, that we may assume the bar at every instant of the passage of the load to be bent into the same curve which it would assume if the point of application of the load were pressed down statically to the same position.*

Thus, in fig. 2, Plate V., let $A E$ be the fixed points upon which the bar is supported, and let the dotted curve $A\, b_2\, c_3\, d_4\, f g$ be the trajectory which the body describes in its passage along the bar with considerable velocity. Draw through the points $A\, b_2\, E$ the curve $A\, b_2\, c_2\, d_2\, E$, into which the bar would be bent, if a sufficient weight were suspended at b_2, to depress the bar to that point. This curve may be supposed to be the form into which the bar is actually thrown at the instant of the body's passage over the point b_2 of the trajectory.† Similarly, when the body passes over the central line at c_3, the momentary form of the bar will be obtained by drawing through the points $A\, c_3\, E$ the proper curve $A\, b_3\, c_3\, d_3\, E$; and when the body has arrived at d_4, the

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$A, B = \frac{a^2}{8\,S}$ (measured downwards). Its central radius of curvature (measured upwards) is $\frac{a^2}{4\,S}$, or twice the former. The latter, supposing the deflection small, is half the radius of a circle drawn through the extremities of the bar and its central depressed point. The two values of x, which correspond to the two points of contrary flexure, are, $a \pm \frac{a}{\sqrt{3}}$, and the corresponding value of the ordinates is $\frac{4}{9}\,S$.

* This would not be the case if the bar were exceedingly slender, and may perhaps not be strictly true even in some of the experiments given above. I have shown below how this point may be examined, but I do not believe that any sensible error has been introduced into the result by the above assumption.

† The curve of the bar may be drawn by points from its equation, but more simply by means of a slender straight steel rod resting on two pins driven into the drawing-board at the ends of the curve of the trajectory, and depressed by hand to any desired point of the latter.

form of the bar will be $A\ b_4\ c_4\ d_4\ E$. This diagram may serve to illustrate the general nature of the action that takes place in all the experiments in question, and to show how completely different the curve of the trajectory is from the curves into which the bar is bent.

In the equilibrium curve the greatest deflection corresponds to the greatest deflection of the bar, and happens at the centre of the bar, where the two curves have a common tangent. But the above figure shows that this is not the case in the other phases of the trajectory. The point of greatest deflection of the trajectory lies a little beyond d_4. The point where the body produces the greatest central deflection of the bar will be found beyond d_4, by drawing through $A\ E$ a curve of the bar that will touch the trajectory. The entire bar will thus be evidently a little more depressed than the lowest curve shown in the figure.

The operation of the registering apparatus (see page 436) will now be more clearly understood. Five pencils were in reality attached to the bar, but, for simplicity sake, we will suppose only three to have been employed, and fixed to the bar at equal distances, from the ends and from each other respectively, at the points $B\ C\ D$, fig. 2. If these pencils were to trace their lines upon a fixed board, we should merely obtain for each a line that would give the greatest deflection that each point of the bar had attained, but no information with respect to the position of the body at which this greatest deflection was given, or with respect to the trajectory of the body.

In fig. 3, Plate V., the curves of the trajectory and bar are drawn in exact correspondence with fig. 2. The board, placed behind the bar, is supposed to receive a small constant horizontal motion, such that during the passage of the body from A to E the board shall travel through a space equal to the distance from 1 to 5 in the groups of parallel lines shown in the figure opposite to each of the points A, B, C, D, and E.

Thus, at the beginning of the motion, the point A_1 was opposite that end of the bar, and the points $B_1\ C_1\ D_1$ were similarly opposite to the respective pencils with which the bar is furnished. When the body reaches B, the motion of the board brings all the points marked 2 opposite their respective pencils, and when it has reached C, all the points marked 3 will be opposite their respective pencils, and so on. The lines at E similarly show the points of the drawing-board that are brought opposite to that extremity of the bar by the motion. The vertical lines $B\,b_2\,b_3\,b_4$, $C\,c_2\,c_3\,c_4$, $D\,d_2\,d_3\,d_4$, shown in fig. 2, are thus, by the motion of the board, opened out into the curves designated in fig. 3 by the same letters respectively; and these curves furnish as many points through which to draw, not only the trajectory, but the curves of the bar.

When the body had arrived at A, the bar was horizontal, and its figure, therefore, passes through the points $A_1\,B_1\,C_1\,D_1\,E_1$. When the body comes to B, every line headed 2 has come opposite to the respective points of the bar, and the intersections of the pencil curves with these lines taken in order, namely, the points $b_2\,c_2\,d_2$, are points through which the bar must at that instant pass. Similarly, the points $A_3\,b_3\,c_3\,d_3\,E_3$ serve to draw the form of the bar when the body passes the centre, and $A_4\,b_4\,c_4\,d_4\,E_4$ is the curve of the bar when the body passes beneath the point D.

Points in the trajectory, on the other hand, are obtained by taking lines from the groups, each headed with a successive number; thus the lines $A_1\,B_2\,C_3\,D_4\,E_5$ will, by their intersections with the pencil curves, give the points required. For when the body was at A, A was opposite to that point of the bar, and is, therefore, a point in the trajectory. When the body reached B, the line 2 on the board was brought opposite to it, and thus b_2 is the next point in the trajectory, and so on.*

To insure the proper working of this contrivance it is necessary that it should be made with great delicacy and care. A perfectly

* It will, of course, be seen that the length of the trajectory thus obtained is greater than the length of the bar, by a quantity equal to the space 1–5, described upon the board. But this elongation is of no consequence, because it does not destroy the proportion between the abscissæ and ordinates of the curve, the velocity of the board being constant. The curves of the bar obtained in this manner are its real curves, and may serve to try whether the form of the bar is really sensibly different from its statical curvature. But the apparatus in question should only be employed when the experiments are conducted on a tolerably large scale with great loads, because the friction and inertia of its parts may seriously interfere with the motion of bar and load when the latter is small. Hence I have not introduced it into my smaller apparatus.

equable travelling motion ought to be given to the drawing-board by clockwork, or rather the pencils should be so arranged as to trace their curves upon the surface of a cylinder, which is perfectly practicable, although I have preferred describing the mechanism as applied to a travelling-board, on account of its greater simplicity. The board is objectionable, because its length, necessarily limited, compels it to be set in motion as soon as possible before the car is started, else it may arrive at the end of its course before the car has completed its journey over the bar. This increases the difficulty of giving it an equable velocity. A cylinder, on the other hand, may continue revolving as long as may be necessary.*

It will easily be seen that, however irregular the motion of the board may be, a true form of the bar will be always obtained from the group of pencil curves, by taking a series of points at the same respective distances from each other as the pencils. By means of these curves, therefore, we may, without reference to the velocity of the board, determine from each experiment not only the maximum deflection that has been given to every one of the five points in succession, but also the contemporaneous deflection of the remaining points.

Thus in fig. 3 the maximum deflection in the central pencil curve is shown to have taken place between the lines 4 and 5, that is, when the travelling load has reached a point beyond the

* The paper cylinder should be fixed below the bar with its axis parallel to it. Each pencil to be attached to the vertical arm of a right-angled bell-crank lever also mounted below the bar upon a horizontal axis at right angles to the direction of the bar, the horizontal arm of the same lever to be connected with the bar above by means of a link rod, jointed to the arm at its lower extremity and to the bar at its upper extremity. Its connection with the bar to be made by forming the link into a branch embracing the bar, each arm of which has a pointed centre-screw, which enters a small hole punched in the side of the bar (see fig. 8, Plate VI.). Thus, when the bar descends, a horizontal motion will be given to the pencil; and as the bar, the pencils, arms, and links, and the axis of the cylinder, lie in one vertical plane, the same revolving cylinder will receive all the curves. But the apparatus must be carefully constructed, so as to be as light and as free from friction as possible. The pencils should be fixed in small swing frames, and the whole mechanism be protected by a shield between itself and the bar, to avoid injury when the bar breaks.

centre between D and E; and if we take a point upon each curve at the same distance between their respective lines 4 and 5, we shall obtain the deflections at each point respectively that accompanied the maximum deflection at the centre, or, in other words, the form of the bar at that instant. Similarly we might obtain the maximum deflection at B, and the contemporaneous deflections at the other points, and so on for all.

But the form of the *trajectory* of the body can only be determined from such curves when the board moves uniformly, or at least when its motion is perfectly known, and the times of the body passing the several points of the bar registered upon it. As this was found impracticable with the Portsmouth apparatus, from the roughness of the mechanism, and a better mode had presented itself for obtaining the trajectory, the apparatus in question was confined to obtaining the maximum deflection, as above explained.

After all, however, the method of registering the trajectory by five points is evidently insufficient, and for the perfect knowledge of the effects I soon found it necessary that the entire course of the curve should be recorded. This may be effected by causing a pencil attached to the centre of the car to trace a line upon a drawing-board fixed parallel to its course. But this simple expedient can only succeed when the car moves with great steadiness,—a condition which the nature of the Portsmouth apparatus placed wholly out of the question.

The theoretical investigation of the problem is replete with difficulty, and its complete solution appears beyond the bounds of analysis. A limited solution can only be obtained by reducing the conditions to their simplest form, namely, by supposing the weight of the bar to be so small, compared with that of the load, that its mass may be wholly neglected; by considering the load as resting on the bar at one point only, and its mass to be concentrated in that point; and lastly, by supposing the deflection to be small compared with the length, which latter condition is true in practice. With these limitations not only can the form of the trajectory be obtained theoretically, but, as we shall see, other laws can be deduced which completely enable us to group the experimental phenomena and extend them to practical cases.

But for this purpose an apparatus must be so arranged as to approach, as nearly as possible, to the simple conditions upon which the theory is based, in order the better to compare their respective results.

The simplest considerations serve to show that, provided the due proportions be maintained between the loads, velocities, and stiffness of the bar, the curves of the trajectory and bar respectively will be the same, whether small weights running on light bars be employed or heavy loads travelling upon massive bars. But in the former case the experiments may be made with an apparatus capable of construction with any required degree of delicacy and accuracy, with small friction, easily manageable and capable of being contained in an ordinary laboratory; and in the latter case the great loads and heavy bars are necessarily accompanied with unsteadiness of motion and great friction, and a general magnitude and roughness, which makes it necessary to employ several workmen and much time in each experiment, and to require the resources and space of a Government dockyard.

The radical defect of the Portsmouth apparatus, for the purpose we are now seeking, proved to be the employment of a car resting with four wheels upon two trial bars at once. In the first place the load presses with two wheels upon each bar, the bar being 9 feet long and the wheels (or rather axles) 2 feet 10 inches apart; it therefore results that when the car first enters upon the bar, the pressure of the fore wheel only acts upon the latter. When the car has advanced through a space equal to the distance between the axles, the pressure of the hind wheel also begins to act, and now the bar is subjected to the action of two loads pressing at a constant distance from each other, and this continues until the fore wheel reaches the end of the bar, which is then subjected to the pressure of the hind wheel alone. Thus a complex form of trajectory is obtained which cannot be compared with the theoretical results, and which, after all, is not much nearer to the practical effect of a four-wheel carriage upon a bridge than a load pressing on a single point would be, because the distance between the wheels is so much greater in proportion to the length of the bridge than in the real case. Again, great difficulties are introduced by the simultaneous employment of two bars. Whatever care may be taken in selecting bars, it is next to impossible to find a pair of exactly equal strength, or, if found, to arrange the load on the carriage so that it shall press equally upon both bars and upon both hind and fore wheels. Hence an inevitable inequality in the simultaneous deflections of the bars, which, as the centre of gravity of the load is high, throws greater weight upon one side than on the other during the passage of the car. This, besides disturbing the results, tends to induce lateral oscillations that increase unduly the deflections on either side, and produce anomalies in the general effects. It was this lateral shake which prevented the trajectory from being traced by the continuous motion of a pencil. The principal excellence of the Portsmouth experiments consists in the determination of the effect of velocity upon the breaking weights on a large scale, for which purpose they will be found to give a most valuable and novel collection of facts.

For the purpose of obtaining the trajectory experimentally, I found it necessary to contrive and construct an apparatus in which the required conditions of simplicity should be complied with. The principles of this apparatus I had indeed suggested from the beginning, and was desirous of introducing into the larger machine, but it was thought advisable that the latter should be made to resemble the case of a car running on a bridge as much as possible, in order to insure the confidence of practical engineers in the results that might be obtained.

As the purpose of this small apparatus was to determine the trajectory without reference to the fracture of the bars, the material I selected was naturally steel, as being the most elastic and free from set. Thus the same bar could be used for many experiments, which greatly facilitates their comparison. Experiments upon cast iron are always embarrassed by the accumulation of set and the occasional fracture of the bars. The machine was therefore arranged to operate upon steel bars of 4 feet or less in length, and of such a stiffness as would require a weight not greater than 6 lbs. to produce a sufficient deflection.

A single trial bar was employed, and the weight pressed upon that bar at one point only. The arrangement by which these conditions were carried out consists of a carriage, which runs on four wheels, upon a kind of railway. The carriage supports a horizontal swing frame, one end of which is hinged to it; the other end has a roller, which rests on the bar, and is also capable of being loaded at pleasure, so as to press more or less upon the bar. The trial bar, in fact, forms the continuation of an intermediate rail which lies between the two rails that support the wheels of the carriage. Thus the only purport of the carriage is to give steadiness to the weight, and confine its motion to a vertical plane. The weight presses with perfect freedom upon the bar, deflecting it during its passage, while the carriage runs steadily along the horizontal rails between which the bar is fixed. A pencil, attached to the swing frame, rises and falls proportionally to the deflection, and traces the curve of the trajectory upon a vertical drawing-board, which is fixed parallel to the trial bar, and opposite to it.

This apparatus is figured in Plate VI., and I will now proceed to describe its details.

Figs. 1 and 2 show the plan and elevation of the railway, and its inclined plane.

Figs. 3 and 4 show, on a larger scale, the central part of the railway at the place where the trial bar is fixed, and also the carriage, tracing-point, drawing-board, &c., in detail.

Figs. 5 to 8 exhibit lesser details of the mechanism.

The frame (*A A*, figs. 3 and 4) of the carriage is a simple rectangle, formed of two longitudinal bars, connected by bolts which pass through two transverse bars. The four wheels of the carriage are fixed to their axles in the manner of railway carriages, but the two axles run between pointed steel centre-screws, to reduce the friction to the least possible. These screws are seen at *D D D D*, fig. 3. The wheels have their flanges turned outwards, contrary to the usual mode. This enables the carriage to run upon a single plank of the proper breadth, and having its edges slightly rounded. The flanges are also thus kept out of the way of other portions of the mechanism in those parts of the fixed frame in which parallel bars are substituted for the plank.

The swing frame is made of thin plate iron, with cross braces, arranged so as to give it as much stiffness and lightness as possible. Its axis, *B B*, is mounted between centre-screws, *E E*, and at the other end it carries a roller, *G*, which rests upon the trial bar. Leaden weights, *H*, can be fixed in any number to this end of the swing frame, by means of a thumb-screw; and a small stage, the end of which is seen in fig. 4, is provided to support them.

In fig. 4, the trial bar, *I K*, is shown, and the carriage is represented in the act of passing over it. The wheels of the carriage run upon the side rails of the fixed frame, or tramway. The swing frame, however, is sustained at the front end by the carriage, and at the hinder or heavy end it rests upon the trial bar, by means of the roller, and depresses it during its passage.

The weights being fixed between the roller and the axis of the swing frame, produce less pressure on the bar than their actual weight. This pressure, however, can be accurately measured by a spring dynamometer, applied to the axis of the roller.

The axis of the swing frame is placed as low as it can be, without touching the frame and trial bar in its passage. The centre of the roller, therefore, describes in its short motion an arc of a circle, which differs but little from a vertical line with respect to the frame of the carriage; for the radius of the swing frame is 20 inches, and the total vertical motion of the roller never greater than 2 inches.

The general arrangement of the tramway is shown in figs. 1 and 2. A plank, *O P*, set at an angle of 45° with the horizon, rests at the upper end, *O*, against the wall of the room, and at the lower end, *P*, upon a triple frame, *P Q R S*. The two outer portions of this frame are exactly similar. The upper edge of each, from *P* to *Q*, is straight, and inclined in continuation of the plank; and from *R* to *S* is straight and horizontal. The edge from *Q* to *R* is an arc of a circle of 7 feet radius, which touches the inclined edge at one extremity and the horizontal edge at the other, so as to connect the inclined line with the horizontal.

The frames are set at such a distance from each other as will allow the carriage-wheels to run upon their upper edges, like a railway, with as little lateral shake as possible; and the plank is carefully made of the same breadth. Thus if the carriage be set upon the plank, and released, it will run down it, and be conducted by means of the curved portion upon the horizontal rails.

The roller of the swing frame at first simply rests upon the plank; but when it passes beyond the lower end of the plank, a support for it is supplied by the intermediate frame, *P I*, seen in the plan, fig. 1. This frame has a similar straight edge, *P Q*, and a curve, *Q R*, to the outer frames between which it is fixed. But as the trial bar *I K* is higher than the edges of the tramway, for the convenience of better access to it, the curve *Q R* is arranged to conduct the inclined part to this higher level; and accordingly, in the elevation, fig. 2, the intermediate curve is seen rising above the lateral curves, and thus ending below with a horizontal tangent, *R I*, higher by an inch and a half than the lateral rails.

The lateral rails, as already explained, are continued as far as *S*; but the middle rail is cut off at *I*, and the brass chair, or contrivance for holding the trial bar, is fixed to the end of it.

The trial bar, *I K*, thus forms the continuation of the middle rail; and the loaded roller of the swing frame thus runs from the middle rail to the trial bar. At the far end, *K*, of the trial bar, a second chair is attached to a rail, *K N*, which receives the roller after it has passed over the trial bar. To adapt the apparatus to receive bars of different lengths, this latter rail can be shifted in position. Its extremity, *K*, terminates in a flat square piece, which is rebated beneath, so as to rest upon and lie between the upper inner edges of the side rails. A similar piece of wood, *Y*, is rebated to slide between the lower inner edges of the side rails; and a bolt and thumb-screw, passing through the whole, serves to fix the end, *K*, of the shifting rail, at any distance from the other rail, *I*, that will suit the bar in question. The shifting rail is sloped gradually downwards from *K* to *N*, so that the roller of the swing frame gradually sinks downwards in its passage until it is caught by a stop in the carriage, after which the middle rail is no longer required to sustain it. *Z* is a hook-bolt, which serves to fix the middle of the shifting rail.

When the carriage has passed off the trial bar, it is necessary to check its motion and bring it to rest. To effect this, two boards, *T V*, *T V*, are fixed in continuation of the tramway. These boards are fixed at a greater interval than the tramway, and are also slightly inclined upwards, and divergent. Their interval is adjusted so that the side bars of the carriage may rest upon them, as shown by the carriage in the figure.

When the fore wheels of the carriage have nearly reached the point *T*, the lower surfaces of its frame touch the slightly inclined edges of the boards *T V*, between *T* and *S*. Thus the frame is gently lifted, so as to raise the wheels from the railway; and as the carriage proceeds it is converted into a sledge, of which the boards *T V* form the sledgeway. But as the friction of this sledge is by no means sufficient to stop the carriage, four springs (marked *check-springs* in figs. 4 and 5, and also shown in the small figures of the carriage at each end of the figs. 1 and 2) are screwed to its sides. These springs stand completely free so long as the carriage runs on the railway; but immediately after the carriage has become shifted to the sledgeway, the springs begin to press upon the sides of the latter, which are, as the plan shows, divergent; and the divergency is greater at the beginning, *T*, because the boards are planed to a thin edge to increase it. Thus the pressure of the springs gradually increases as the carriage proceeds along the sledgeway. They are made of sufficient strength to stop the carriage before it reaches *V*, when it is released from the top of the inclined plane.

The whole of the frame above described is fixed together by bolts, which pass through the legs, the side frames, and intermediate blocks, so as to allow the whole to be readily taken to pieces, or remounted at pleasure. At *W*, a transverse frame, consisting of a horizontal piece below, with two legs, and with a sloping brace rising to the height of the vertical rail, to which it is bolted, serves to give lateral support to the whole machine.

The side rails are divided at the leg near *I*. This reduces the size of the parts of the frame, and also allows longer rails to be substituted from *I* to *S*, when longer trial bars are required.* The machine represented in the drawings will not receive bars longer than 4 feet. For the purpose of conveniently raising the carriage, and releasing it, a pulley, *O*, is fixed to the upper end of

* The frame is farther secured to the floor by a bolt near *W*, the nut of which bears upon a short transverse piece laid upon the horizontal rails, close to the upright post. This is necessary, to enable it to sustain the plank, which plank is also prevented from sagging by a brace, as shown. But nearly the whole of the phenomena of the experiments may be sufficiently shown by a less velocity than that acquired from the top of the plank, namely, 30 feet per second. About 20 feet per second will be found amply sufficient for repeating these experiments, if desired, and a plane extending about 4 feet above *P* will therefore be enough. The construction of the inclined part of the framework may thus be simplified by making the straight

the plank. The cord which passes over this is attached to a small sledge, *a*, figs. 1 and 2, upon which is fixed a latch and detent. The latch is adapted to receive a hooked pin (*n*, figs. 3 and 4), fixed to the end of the carriage; and when the detent is in the position shown in fig. 1, the carriage is thus united to the sledge, and can be drawn up with it to any desired altitude of the plane by means of the cord, and secured there. But the string, *b*, fig. 2, passes over the pulley, *c*, fixed to the little sledge, and is tied to the detent. Pulling this string, therefore, the detent is shifted, and the latch releases the carriage, which then runs down the inclined plane, and passes over the trial bar.

Fig. 5 represents the mode in which that end of the trial bar which first receives the action of the roller is fixed.

The extremity of the intermediate rail bar of the frame is cut vertically from *a* to *c*, and has a horizontal step, *c d*.

e f is a piece of metal or chair, which is secured against the vertical face, *b a c*, by means of a screw-bolt, *g*, the nut, *h*, of which is inserted into a mortise in the rail. The screw passes through a mortise in the metal piece, and the latter is kept in a vertical position by a shallow grooved recess, sunk in the vertical face, *a b c*, of the rail. Thus the chair admits of a vertical adjustment of its position. A capstan-headed screw is tapped into its lower extremity, and the head of this screw rests upon the step, *d*, which has been already mentioned. By slightly releasing the screw *g*, and turning the capstan head to right or left, the vertical adjustment is made at pleasure.

The upper end, *e*, of the metal chair has a square notch cut in it, and a steel centre-screw on each side. The points of these screws are received into corresponding centre-punch holes at the end of the trial bar, which is thus held in a manner that admits of free vertical deflection of the bar.

It is essential that the roller, as it first comes upon the bar,

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portion down to *Q* of the plank form, and sustaining the whole on its legs, without employing the heavy plank resting against the wall. In exhibiting the experiments to an audience, it is convenient to connect the centre of the bar with an index, contrived so as to magnify its deflection four or five times; thus the increase of deflection produced by velocity is shown very clearly.

should meet with no inequality of level that would either jerk it upwards or let it drop and rebound from the bar.

To effect the smooth entrance required, the vertical adjustment just described is provided, and it will be seen in the figure that the end of the bar also projects into a sunk recess formed upon the upper face of the rail. When the vertical adjustment is properly made, the upper face of the rail and the upper surface of the bar are made to coincide in level, and as the roller is sufficiently broad to run upon the sides of the above-mentioned recess, it is thus gradually brought upon the bar, the extreme end of which is slightly lowered by the file, to facilitate this action.

This chair, being attached by the single bolt, *g*, can be readily removed from the frame, to substitute others of different forms, if required for differently shaped bars.

The far extremity, *K*, of the bar is supported by the contrivance shown in fig. 6, which represents the end, *K*, of the shifting rail. When the roller has passed completely over the bar, there is no necessity to provide for its level exit, as for its level entrance, for the work has been completed at this point. All that is wanted is to support it beneath in such a manner as will allow it to slide out a little, because when it is bent by the deflection of the weight the end of it is necessarily slightly drawn out of its recess in this farthest chair. The first chair grasps its end of the bar by centre-points, as we have seen, so as to prevent this drawing action at the beginning, where it would be mischievous, and it is so small at the other end that it is not worth while to provide a friction-roller or such contrivance. The farthest end of the bar is therefore allowed to rest in a grooved piece of metal, *a a b*, the groove of which is made rather wider than the widest bar employed, and a pair of blunt-ended screws, *c c*, serve to keep the bar steady laterally, being screwed up so as just to touch without pinching it. We shall presently see that the action of the weight tends to make the bar fly upwards when it reaches the end of its course. To keep it in its groove, therefore, a steel stirrup, *d e*, is provided; this is adjusted so as just not to touch the top of the bar, and the bar itself is filed into such a curve on its upper side as will enable it to escape contact with this stirrup during its deflection. The diagram, fig. 7, will explain this, in which $a\ b$ is the bottom of the groove, e the section of the stirrup, $d\ b\ a$ the end of the bar, the upper face of which is filed into a curve, as shown. The dotted line shows the position, greatly exaggerated, into which this end is thrown by the sliding motion which accompanies its deflection; and also shows how the curve enables it to escape the stirrup.

In figs. 4 and 5 a wedge-shaped piece, Z, is shown attached to the end of the shifting rail. This is for the purpose of receiving the roller of the swing frame, if the bar should break. In this case the swing frame would drop downwards until it rested upon its stage in the carriage, and its roller would meet the wedge, Z, and be conducted to the upper face of the shifting rail, and thus prevented from stopping the carriage suddenly or throwing it off.

The board which receives the trace of the trajectory is shown at $L\ M$, in figs. 1, 2, 3, 4. It is sustained upon two iron pillars, screwed below to the side rails, and above to the back of the board. These pillars are curved outwards, so as to escape the heads of the centre-screws of the carriage, which would otherwise strike them in their passage. The board is thinned away at each end, as the plan, fig. 3, shows. Thus the front pencil, which is carried by the swing frame, encounters a gently-sloping surface at its first contact with the paper, and is also gradually released as it quits the paper. To fix the paper, its extremity must be first grasped in the wooden clamp at M, then stretched tightly along the surface of the board, and doubled over the end, L; a small iron clamp may then be applied, and will be found sufficient to hold it. It is better to apply a third clamp at the middle, to prevent it from sagging. The best paper that I have tried is that which is prepared by Messrs. Harwood, of Fenchurch Street. This will receive the trace of a pointed brass wire. It can be had in any length required, and should be mounted upon calico. The softness of the calico enables the pencil to act better during its rapid motion, and also allows the paper to be stretched tighter without fear of tearing.

The swing frame has a piece, h, figs. 3 and 4, attached to its side, as near to the roller as the wheel will allow. The pencil is clamped in a socket at the top of a small triangular swing frame, $k\ k$, which revolves upon centre-points, tapped into its lower extremities, and resting in holes punched in the sides of the piece, *h*, as shown. This piece, *h*, is horizontal in the part shown in the plan, fig. 3, but is turned vertically upwards to receive these centre-screws, as seen in fig. 4. It carries also an upright post, *i*, to which is screwed a wire fork, upon whose branches is strained a ring of vulcanized caoutchouc, *m m*, which is thus stretched into the form of a double horizontal elastic strap, against which the pencil frame is pressed. A silk string fastened to this frame is wound round a fiddle peg, turning with stiff friction in a hole at the top of the post, *i*; by turning this peg to the right or left, the string is tightened or relaxed, and the swing frame pressed more or less against the elastic strap. Thus the pressure upon the pencil can be adjusted at pleasure, its point being, of course, set further outwards in the socket if the frame be drawn more backwards, and *vice versâ*. Also the string limits the outward portion of the pencil so as to insure that it shall touch the sloped part of the drawing-board exactly at the proper point for the first contact.

When the paper and pencil are properly adjusted, the first thing to be done is to conduct the carriage as slowly as possible from one end of the trial bar to the other; the pencil will then trace the *equilibrium trajectory*, as shown by the close dotted line in the figure. This trajectory serves as a curve of comparison for the dynamical trajectory, and a straight line drawn through its level extremities enables us to measure the central statical deflection. The carriage may now be drawn up the inclined plane and released. The pencil will now be found to trace a curve somewhat similar to the interrupted dotted line in the figure; this is the *dynamical trajectory*.

With respect to all the trajectories drawn by the apparatus, it must be recollected that the pencil-point is considerably above the level of the axis of the swing frame, and that its radial distance from that axis is less than that of the roller. Both these causes tend to alter the form of the trajectory, but may be corrected as follows:—

The pencil describes a short arc of a circle, which, like that of the roller's motion, coincides so nearly with a straight line, that it may be considered as one, but as a line inclined to the vertical 13° by the difference of level between the pencil-point and the axis. In transferring the curve, therefore, a sufficient number of ordinates must be drawn on the original papers, inclined 13°, and their length must be transferred to vertical ordinates on another paper to obtain the true curve.

If the form of the curve only be required, the abscissæ of the copy may have any convenient proportion to those of the original; and accordingly, to exhibit the forms of the trajectories more strikingly, I have in Plate VIII. reduced the abscissæ to about one-fifth of the originals, and transferred to them vertically the actual lengths of the original sloped ordinates, as the most rapid and convenient mode of at once reducing the four-feet length of the original trajectories to a commodious size, and of exhibiting the required exaggeration of the form. But if the actual trajectories are required, the length of these sloped ordinates must be increased in the proportion of the radial distance of the roller from the axis of the swing frame to that of the pencil, namely, in the actual machine, of 20 inches to 17·5 inches.

The velocity of the carriage may be nearly estimated by the altitude of the point whence its centre of gravity has been liberated on the inclined plane above the position of that centre on the horizontal rails; but as some loss of this velocity is occasioned by the friction of the wheels and their rotation, &c., some method of measuring the velocity after it has passed over the rail is required. Now, immediately after this passage we have seen that the end of the carriage is received on an inclined sledgeway, and the fore wheels suddenly lifted off their rails. This happens before the check-springs have touched the sides of the sledgeway, and therefore before they have acted to retard its motion. Hence the fore wheels revolving free of the rails, their circumferences retain a velocity equal to that with which the carriage was progressing when the wheels were lifted. By observing, therefore, the velocity of rotation of the wheels when the carriage is checked, we can estimate the velocity with which it had passed over the trial bar. To facilitate this, a worm is formed upon the axis of the fore wheels, and a toothed disk, *C*, fig. 3, loosely geared into it.* An observer, stationed at a point where the carriage stops, with a stop-watch, can easily measure

* This is omitted in the section, fig. 4, to prevent confusion.

the time occupied by the passage of 10 or 20 teeth, and hence obtain the required velocity of rotation. There is, in fact, very little loss of velocity from the retarding causes. The weight of the entire carriage and its mechanism, when the swing frame is loaded to produce a pressure of 4 lbs., is 28 lbs. I find that when the carriage is released from a height that would generate, without retarding causes, a velocity of 10 feet per second on the horizontal rails, the actual velocity ascertained by the above method is 7·7 feet. Similarly, velocities that should be 15 and 20 feet, are respectively reduced to 12 feet and 16·6 feet.

In fig. 3, two centre-screws, *F F*, will be observed in the sides of the frame, supporting the axis of an arm, which is shown in dotted lines only, and terminates in a roller, *G'*, which rests, like the roller, *G*, of the swing frame, upon the trial bar, but at a distance of one foot behind it. This arm and roller is also dotted into fig. 4. Their purpose was to obtain the equilibrium and dynamical trajectories in the case of two equal pressures acting upon the trial bar, as in the Portsmouth experiments: want of time, however, having prevented me from obtaining accurate results with this part of the apparatus, I have contented myself with inserting the arm in the drawings by way of suggestion to future observers.

Beneath the bar, in fig. 4, is a contrivance termed the *Inertial Balance*. This will be fully described in Chapter IV. Figure 8 also belongs to this part of the machine.

Trajectories drawn by the apparatus above described are given in Plate VIII.; but the consideration of them is so much involved in the question of the inertia of the bar which our theoretical investigations suppose so small as to be neglected, that I must postpone their explanation until I have given, first, the theory on the above hypothesis, and next, the explanation of the methods by which the inertia of the bar can be introduced.

CHAPTER III.

Theoretical Investigation of the Trajectory.

To simplify as much as possible the mathematical calculation, the carriage must be considered as a heavy particle, and the inertia of the bar neglected. Let $x\ y$ be the co-ordinates of the moving body, x being measured horizontally from the beginning of the bar and y vertically downwards, M the mass of the body, V its velocity on entering the bar, $2\,a$ the length of the bar, g the force of gravity, S the central statical deflection, that is to say, the deflection that is produced in the bar by the body placed at rest upon its central point, R the reaction between the body and the bar. The deflection is small,* and therefore this reaction may be supposed to act vertically, for it must be recollected that the reaction is perpendicular to the curve of the *bar* and not to the *trajectory*, and therefore, in the case of such small deflections as we have to deal with, the horizontal component of the reaction will be insignificant. Thus the horizontal velocity V will remain constant during the passage of the body along the bar. Now we have seen (pp. 453, 454) that a given weight W, suspended to the bar at a distance x from its extremity, will produce a deflection $y = c\,W\,(2\,a\,x - x^2)^2$, c being a constant

* Practically, the deflection of a girder is so small compared with the length, that this hypothesis may be fairly assumed. Engineers inform us that a deflection from $\frac{1}{400}$ to $\frac{1}{600}$ of the length may be allowed in a girder (*vide* Report, Analysis of Evidence, art. *Deflection of Girders, &c.*); but the deflections with ordinary loads are not greater than one-fourth of these. Thus, in Mr. Hawkshaw's evidence (No. 152), we find a deflection of half an inch assigned to a girder-bridge of 89 feet span under the action of a heavy locomotive engine. This is only $\frac{1}{2136}$ of the length; and in the experiments of the Commission at Ewell and Godstone, deflections of $\frac{1}{2320}$ and $\frac{1}{1440}$ of the length were obtained from a heavy locomotive and tender. In the experiments at Portsmouth, on 9-feet bars, deflections of 5 inches, that is, of $\frac{1}{21}$ of the length, were sometimes reached; but even these may be called small in the mathematical sense.

depending on the elasticity and transverse section of the bar. But as the inertia of the bar is neglected, its elastic reaction upon the travelling weight will be equal to a weight that would, if suspended to the bar at a point where the travelling weight touches it, depress that point to the same amount below the horizontal line. Therefore, $R = W = \frac{y}{c} \frac{1}{(2\,a\,x - x^2)^2}$. The constant c may be determined by observing that if $R = M\,g$ and $x = a$, y becomes S. Whence, substituting in the above equation, we obtain $c = \frac{S}{M\,g\,.\,a^4}$.

The forces which act on the body are its gravity and the reaction of the bar. Whence we obtain the equation of motion,

$$\frac{d^2 y}{d\,t^2} = g - \frac{g\,a^4}{S} \times \frac{y}{(2\,a\,x - x^2)^2}$$

which becomes, since $V = \frac{d\,x}{d\,t}$,

$$\frac{d^2 y}{d\,x^2} = \frac{g}{V^2} - \frac{g\,a^4}{V^2 S} \frac{y}{(2\,a\,x - x^2)^2}:$$

from the integration of this equation we should obtain the curve of the trajectory.

Having proceeded thus far, however, I found the discussion of this equation involved in so much difficulty, that I was compelled to request my friend G. G. Stokes, Esq., Fellow of Pembroke College,* to undertake the development of it. His kind and ready compliance with my wishes, and his well-known powers of analysis, have produced a most valuable and complete discussion of the equation in question. The mathematical methods employed for this purpose are, from their nature, probably unintelligible to the majority of practical men, for whom the present essay was written; and it was thought better, therefore, that the discussion should be thrown into the form of a paper, and presented to the Cambridge Philosophical Society, before which it was read the 21st May, 1849:† to the Transactions of that

* Now Lucasian Professor in the University of Cambridge.

† The title of the paper is as follows: 'Discussion of a Differential Equation relating to the Breaking of Railway Bridges,' by G. G. Stokes, M.A., Fellow of Pembroke College, Cambridge.—*Transactions of the Cambridge Philosophical Society*, vol. viii. page 707. 1849.

Society I must beg to refer those of my readers who may desire to follow out this most elaborate and able investigation. I shall, however, give his results, extracting from the paper such of his remarks as may be necessary to make them intelligible, and shall then proceed to compare them with the trajectorial curves of my apparatus and with practice.

It appears that the equation cannot be integrated in finite terms, except for an infinite number of particular values of a certain constant involved in it; but Mr. Stokes has investigated rapidly convergent series, whereby numerical results may be obtained. By merely altering the scale of the abscissa and ordinates, the differential equation is reduced to one containing a single constant, which he terms β. This he effects as follows:—
Put

$$x = 2\,a\,X \qquad y = 16\,S\,Y \qquad \frac{g\,a^4}{V^2\,S} = 4\,a^2\,\beta\,;$$

and substituting these values in the equation, it becomes

$$\frac{d^2\,Y}{d\,X^2} = \beta - \frac{\beta\,Y}{X - X^2)^2}.$$

"It is to be observed that X denotes the ratio of the distance of the body from the beginning of the bar to the length of the bar; Y denotes a quantity from which the depth of the body below the horizontal plane in which it was at first moving may be obtained by multiplying by $16\ S$; and β, on the value of which depends the form of the body's path, is a constant defined by the equation $\beta = \frac{g\,a^2}{4\,V^2\,S}$. A small value of β, therefore, corresponds to a high velocity, and a large value to a small velocity. It appears, from the solution of the differential equation, that the trajectory of the body is unsymmetrical with respect to the centre of the bridge, the maximum depression of the body occurring beyond the centre. The character of the motion depends materially on the numerical value of β. When β is not greater than $\frac{1}{4}$, the tangent to the trajectory becomes more and more inclined to the horizontal, beyond the maximum ordinate, till the body gets to the second extremity of the bridge, when the tangent becomes vertical. At the same time the expressions for the central deflection and for the tendency of the bridge to break become infinite. When β is greater than $\frac{1}{4}$, the analytical expression for the ordinate of the body at last becomes negative, and afterwards changes an infinite number of times from negative to positive, and from positive to negative. The expression for the reaction becomes negative at the same time with the ordinate, so that, in fact, the *body leaps*. The occurrence of these infinite quantities indicates one of two things; either the deflection really becomes very large, after which of course we are no longer at liberty to neglect its square, or else the effect of the inertia of the bridge is really important. Since the deflection does not really become very great, as appears from experiment, we are led to conclude that the effect of the inertia is not insignificant; and, in fact, I have shown that the value of the expression for the *vis viva* neglected at last becomes infinite. Hence, however light be the bridge, the mode of approximation adopted ceases to be legitimate before the body reaches the second extremity of the bridge, although it may be sufficiently accurate for the greater part of the body's course."

We shall presently see that in practice β is never less than $\frac{1}{4}$, and that the above conclusion can be perfectly reconciled with the experimental results when the inertia of the bar is taken into account. For the investigation of the series by which our author was enabled to calculate the numerical results, I must refer to his paper, from which I have extracted the two following Tables (V. and VI.), which contain a sufficient number of ordinates to enable the trajectory to be laid down by points, in the forms corresponding to nine values of β. Those which belong to intermediate values of β can be easily interpolated. The curves themselves are carefully laid down in Plate VII., fig. 4.

TABLE V.

x.	$\frac{y}{S}$, $\beta =$				z, $\beta =$				T, $\beta =$			
	$\frac{5}{36}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{5}{4}$	$\frac{5}{36}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{5}{4}$	$\frac{5}{36}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{5}{4}$
·00	·000	·000	·000	·000	·065	·111	·200	·385	·000	·000	·000	·000
·02	·000	·001	·001	·002	·067	·115	·208	·398	·005	·009	·016	·031
·04	·002	·003	·005	·010	·070	·120	·216	·412	·011	·018	·033	·063
·06	·004	·006	·011	·022	·073	·125	·224	·426	·017	·028	·051	·096
·08	·007	·011	·020	·038	·076	·130	·233	·441	·022	·038	·069	·130
·10	·010	·018	·032	·059	·080	·136	·243	·457	·029	·049	·087	·165
·12	·015	·025	·045	·085	·083	·142	·253	·474	·035	·060	·107	·200
·14	·020	·034	·061	·114	·087	·148	·264	·493	·042	·071	·127	·237
·16	·026	·045	·080	·148	·091	·155	·276	·512	·049	·083	·148	·275
·18	·033	·057	·100	·185	·096	·162	·288	·532	·056	·096	·170	·314
·20	·041	·070	·124	·227	·100	·170	·302	·553	·064	·109	·193	·354
·22	·050	·084	·149	·272	·105	·179	·316	·576	·072	·123	·217	·396
·24	·059	·100	·176	·320	·111	·188	·331	·601	·081	·137	·242	·438
·26	·069	·117	·206	·371	·117	·198	·348	·627	·090	·152	·267	·482
·28	·080	·135	·239	·418	·122	·208	·367	·642	·099	·168	·296	·519
·30	·092	·155	·272	·477	·130	·220	·386	·676	·109	·185	·324	·568
·32	·104	·176	·308	·534	·138	·232	·406	·705	·120	·202	·354	·614
·34	·118	·198	·346	·605	·146	·246	·429	·751	·131	·221	·386	·674
·36	·132	·222	·386	·665	·155	·261	·454	·783	·143	·241	·419	·721
·38	·150	·240	·427	·736	·165	·277	·480	·829	·155	·261	·453	·781
·40	·162	·272	·470	·802	·176	·295	·509	·870	·169	·283	·489	·835
·42	·178	·298	·513	·869	·188	·314	·541	·916	·182	·306	·527	·892
·44	·195	·326	·560	·939	·201	·336	·576	·966	·198	·331	·568	·951
·46	·213	·347	·607	1·01	·216	·360	·615	1·02	·214	·358	·611	1·01
·48	·231	·385	·655	1·10	·232	·386	·657	1·08	·231	·385	·656	1·08

TABLE V.—(*continued.*)

x.	$\frac{y}{S}$				z.				T.			
	$\beta =$				$\beta =$				$\beta =$			
	$\frac{5}{36}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{5}{4}$	$\frac{5}{36}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{5}{4}$	$\frac{5}{36}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{5}{4}$
·50	·250	·416	·705	1·14	·250	·416	·705	1·14	·250	·416	·705	1·14
·52	·270	·448	·755	1·21	·271	·449	·758	1·22	·270	·449	·757	1·21
·54	·290	·481	·807	1·28	·294	·487	·817	1·29	·292	·484	·812	1·28
·56	·311	·514	·859	1·34	·320	·529	·884	1·38	·316	·522	·871	1·36
·58	·333	·548	·911	1·40	·350	·578	·959	1·47	·342	·563	·935	1·44
·60	·355	·584	·964	1·46	·385	·633	1·05	1·58	·370	·608	1·00	1·52
·62	·378	·619	1·02	1·51	·425	·697	1·14	1·70	·401	·657	1·08	1·60
·64	·401	·654	1·07	1·55	·472	·771	1·26	1·82	·435	·710	1·16	1·68
·66	·425	·692	1·12	1·59	·527	·858	1·39	1·98	·473	·771	1·25	1·78
·68	·449	·728	1·17	1·62	·592	·961	1·54	2·13	·516	·837	1·34	1·86
·70	·473	·765	1·22	1·64	·671	1·08	1·72	2·32	·563	·910	1·45	1·95
·72	·498	·801	1·26	1·65	·765	1·23	1·94	2·54	·617	·994	1·56	2·05
·74	·523	·830	1·30	1·66	·883	1·40	2·20	2·80	·680	1·08	1·69	2·15
·76	·548	·874	1·34	1·64	1·03	1·64	2·52	3·08	·751	1·20	1·84	2·25
·78	·573	·908	1·38	1·61	1·22	1·93	2·92	3·42	·835	1·32	2·00	2·35
·80	·598	·933	1·40	1·56	1·46	2·30	3·43	3·81	·935	1·46	2·19	2·44
·82	·623	·972	1·42	1·49	1·79	2·79	4·08	4·26	1·05	1·65	2·41	2·52
·84	·647	1·00	1·43	1·39	2·24	3·46	4·96	4·79	1·20	1·86	2·67	2·58
·86	·669	1·00	1·43	1·25	2·88	4·41	6·17	5·41	1·39	2·13	2·97	2·61
·88	·691	1·04	1·41	1·09	3·87	5·84	7·91	6·10	1·63	2·47	3·34	2·58
·90	·708	1·05	1·37	·883	5·47	8·12	10·6	6·81	1·97	2·92	3·80	2·45
·92	·723	1·05	1·30	·630	8·34	12·1	15·0	7·27	2·45	3·57	4·41	2·14
·94	·730	1·00	1·18	·318	14·3	20·3	23·2	6·44	3·25	4·58	5·23	1·45
·96	·699	1·00	·987	−·014	29·6	43·5	41·8	−·600	4·55	6·69	6·42	−·09
·98	·690	·857	·652	−·404	112·	139·	106·	−65·8	8·80	10·9	8·32	−5·16
1·00	0	0	0	0	∞	∞	±∞	±∞	∞	∞	±∞	±∞

TABLE VI.

x.	$\frac{y}{S}$, $\beta =$ 3	5	8	12	20	z. $\beta =$ 3	5	8	12	20	T. $\beta =$ 3	5	8	12	20
·00	0	0	0	0	0	·600	·714	·800	·857	·909	0	0	0	0	0
·05	·023	·027	·030	·032	·034	·640	·755	·835	·886	·931	·122	·143	·159	·168	·177
·10	·089	·103	·113	·119	·123	·689	·798	·872	·915	·950	·248	·287	·314	·330	·342
·15	·195	·220	·237	·246	·252	·751	·846	·910	·945	·970	·383	·431	·464	·482	·495
·20	·327	·367	·389	·399	·405	·799	·897	·950	·975	·989	·511	·574	·608	·624	·633
·25	·486	·535	·558	·565	·572	·863	·951	·991	1·004	1·016	·647	·714	·743	·753	·762
·30	·661	·713	·722	·728	·721	·936	1·010	1·023	1·032	1·023	·786	·849	·859	·867	·859
·35	·843	·888	·889	·877	·859	1·018	1·073	1·074	1·059	1·038	·926	·976	·977	·963	·944
·40	1·023	1·049	1·026	·997	·966	1·110	1·138	1·114	1·081	1·049	1·066	1·092	1·069	1·038	1·007
·45	1·190	1·183	1·127	1·078	1·035	1·214	1·207	1·150	1·099	1·056	1·202	1·195	1·138	1·089	1·046
·50	1·331	1·274	1·180	1·111	1·060	1·331	1·274	1·180	1·111	1·060	1·331	1·274	1·180	1·111	1·060
·55	1·431	1·314	1·179	1·092	1·037	1·461	1·341	1·203	1·114	1·058	1·446	1·327	1·191	1·103	1·047
·60	1·486	1·281	1·108	1·018	·968	1·602	1·390	1·202	1·105	1·051	1·538	1·334	1·154	1·060	1·009
·65	1·446	1·173	·954	·895	·860	1·748	1·417	1·179	1·081	1·038	1·590	1·289	1·072	·983	·945
·70	1·334	·983	·781	·733	·720	1·891	1·393	1·107	1·039	1·021	1·588	1·170	·930	·873	·858
·75	1·111	·716	·564	·554	·570	1·974	1·273	1·003	·984	1·013	1·481	·955	·752	·738	·760
·80	·772	·396	·341	·382	·405	1·885	·968	·832	·932	·989	1·206	·620	·532	·596	·633
·85	·335	·090	·172	·241	·254	1·286	·344	·660	·925	·976	·656	·176	·336	·472	·498
·90	$-$·126	$-$·080	·104	·131	·123	$-$ ·970	$-$ ·616	·802	1·013	·947	$-$ ·349	$-$ ·222	·289	·365	·341
·95	$-$·297	$+$·045	·068	·026	·034	$-$8·227	$+$1·248	1·884	·720	·943	$-$1·563	$+$1·237	·358	·137	·179

In Table V. the length of the bar is divided into 50 parts; but in Table VI. 20 divisions were thought sufficient. Each Table, however, consists of three parts. In the first are contained the values of the ordinates of the curve, S being considered as unity.* In the second part of the Table, which is headed z, we have the numerical values, which express the ratio of the depression of the moving body at any point to its statical depression, that is to say, to its place in the equilibrium trajectory. In the third part, headed T, are the numbers which express the tendency of the bar to break at each point, which were thus obtained.

If a weight, W, be placed on a point of the bar whose distance from the first extremity is x, then, by the known principles of

* The equilibrium trajectory may be laid down by the help of the subjoined Table. The length of the bar is divided into 50 parts, and as the curve is symmetrical on each side of the centre, it is only necessary to give the ordinates for the first half: the central ordinate may be assumed of any convenient magnitude, and divided into 1000 parts.

TABLE VII.—EQUILIBRIUM TRAJECTORY.

x	y	x	y	x	y
1	5	10	410	19	909
2	24	11	476	20	920
3	51	12	532	21	941
4	86	13	589	22	970
5	129	14	655	23	986
6	178	15	707	24	995
7	231	16	753	25	1000
8	285	17	808	—	—
9	344	18	851	—	—

Table VI. contains the results for five values of β, namely, 3, 5, 8, 12, and 20, upon which Mr. Stokes makes the following remarks :—

" The form of these trajectories is shown in fig. 4, Plate VII. As β increases, the first point of intersection of the trajectory with the equilibrium trajectory moves towards A. Since $z = 1$ at this point, we get from the part of the Table headed z, for the abscissæ of the point of intersection (by taking proportional parts) ·34, ·29, ·26, ·24, and ·22, corresponding to the respective values 3, 5, 8, 12, and 20, of β. Beyond this point of intersection the trajectory passes below the equilibrium trajectory, and remains below it during the greater part of the remaining course. As β increases, the trajectory becomes more and more nearly symmetrical with respect to C: when $\beta = 20$ the deviation from symmetry may be considered insensible, except close to statics,* the strain upon this point, or tendency of the bar to break, is measured by W multiplied by the product of the two parts into which the bar is divided by the point upon which the weight rests, or by $W \times (2a - x)\,x$. But, in the problem under consideration, the dynamical action of the travelling load, combined with the elastic reaction of the bar, deflects the point of the bar upon which it is momentarily placed to a distance, y, below the horizontal line. Since, therefore, the inertia of the bar is neglected, the effect to break the bar is the same as if a weight were suspended to this point sufficiently great to depress it statically to the same distance, y. Such a weight is equal to the reaction of the bar, and is therefore proportional to $\frac{y}{(2ax - x^2)^2}$. Substituting this value of W in the above expression, we obtain the tendency of the bar to break under the action of the travelling load proportional to $\frac{y}{2ax - x^2}$. Call this tendency T, and let T be so measured that it may be equal to unity when the moving body is placed at rest on the centre of the bar; in which case $y = S$ and $x = a$.

$$\text{Hence } T : 1 :: \frac{y}{(2ax - x^2)} : \frac{S}{a^2} \text{ and } T = \frac{a^2}{S} \cdot \frac{y}{2ax - x^2}.$$

In this manner, the numbers in the third part of the Tables were obtained. It must be remembered that, in this part of the investigation, the inertia of the bar or bridge is necessarily

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the extremities $A\,B$, where, however, the depression itself is insensible. The greatest depression of the body, as appears from the column which gives y, takes place a little beyond the centre; the point of greatest depression approaches indefinitely to the centre, as β increases. This greatest depression of *the body* must be carefully distinguished from the greatest depression of *the bridge*, which is decidedly larger, and occurs in a different place, and at a different time (see p. 457). The numbers in the columns headed T show that T is a maximum for a value of x, greater than that which renders y a maximum, as in fact immediately follows from a consideration of the mode in which y is derived from T. The first maximum value of T is about 1·59 for $\beta = 3$, 1·33 for $\beta = 5$, 1·19 for $\beta = 8$, 1·11 for $\beta = 12$, and 1·06 for $\beta = 20$."—*Camb. Trans.* p. 723.

* *Vide* Barlow on 'the Strength of Materials,' or any statical writer on this subject.

neglected, and it will be seen below that this inertia greatly affects some of these results.

Having now stated the results of Mr. Stokes's discussion of the equation to the trajectory, I shall endeavour to apply them to the interpretation of the experiments. This discussion has shown that the curve of the trajectory assumes different phases, each of which is characterized by a certain value of the constant $\beta = \frac{g\,a^2}{4\,V^2\,S}$. Their forms are shown in fig. 4, Plate VII. When β is large, the curve departs very little from symmetry, or from the form of the equilibrium trajectory. But, as β becomes smaller, the first half of the curve rises more and more above the equilibrium curve; the second half sinks, on the contrary, below it at first; but when the value of β is less than about $\frac{5}{4}$, the loop of the trajectory begins to rise again. On the other hand, however, as β diminishes, this loop, or lowest point of the curve, steadily increases its distance from the central position which it holds in the equilibrium trajectory.

Every one of these phases or forms of the curve may have its ordinates upon any scale of proportion with respect to the length of the whole. This scale is governed by the proportion of a to S. Accordingly, in the drawings of the curves, the proportional magnitude of the ordinates is assumed much larger than in actual practice, or, indeed, than would be consistent with the hypothesis that the deflections are small compared with the length of the bar.*

But before we can apply these results in illustration of the experiments, we must ascertain the numerical values which β holds in practical cases. In the expression for β, $g = 32{\cdot}2$ feet, a is the half-length of the bridge in feet, V the horizontal constant velocity of the body in feet per second, and S the central statical deflection, also in feet.

It will be more convenient if the value of β be expressed in

* A numerical example may explain the above remarks. In the expression for β $\left(\text{namely } \beta = 24{\cdot}15\ \frac{l^2}{V^2\,S}\right)$ let us substitute the values given in the two following cases. (1.) A bridge 30 feet long, over which a load that would produce a statical deflection of ·22 inch, is travelling at the rate of 90 feet per second. (2.) A bar 9 feet long, on which a load that would produce a statical deflection of 2 inches is travelling at the rate of 9 feet per second.

terms of the length (l) of the bridge, instead of the half-length, and also, if the deflection be expressed in inches, the other quantities, l and V, being expressed in feet. If we make the necessary substitutions for this purpose in the formula, we obtain $\beta = 24{\cdot}15 \frac{l^2}{V^2 S}$.

In the 9-feet bars of the Portsmouth experiments, $\beta = \frac{1956{\cdot}15}{V^2 S}$.

It is clear that, as the velocity and statical deflection vary, every experiment has a different value of β. But as certain selected values of the velocity were employed, we can exhibit corresponding values of β, as in the following Table, in which also a few values of S are taken, between which it is easy to estimate the value of β for any particular case.

TABLE VIII.

S, in Inches.	Velocity in Feet.			
	15	29	36	43
·3	29·0	7·74	5·02	3·54
·6	14·5	3·87	2·51	1·77
1	8·69	2·32	1·51	1·06
1·5	5·79	1·55	1·00	..
2	4·35	1·16	..	..
3	2·89	..	..	..

The values of S in each column are not extended beyond those which were employed in the actual experiments, as shown by the Tables (pp. 443, 488), and it thus appears that β was never less than unity, or greater than 30, in the three first series of these experiments.

To obtain less values of β, we must diminish the length of the

We shall obtain the same value of β for each of these examples, namely, 12, very nearly. The trajectory of each of these will be the same, and also the same as that given for $\beta = 12$ in Plate VII.; in this respect, that the *proportional* increase of the statical deflection at similar points of the length is the same in all three. But the relative scale of the abscissæ and ordinates will be different in every one; for in the bridge, the central statical deflection is to the length as 30 feet to ·22 inch, that is, as 1636 to 1; in the bar the deflection is to the length as 9 feet to 2 inches, or as 54 to 1; and in the figures on the Plate as 10 to 1.

bar, or employ greater velocities and larger statical deflections; that is to say, greater weights. But greater velocities are not to be obtained with the inclined plane, which was already carried as high as practical limits allowed; and larger proportional deflections would remove the case beyond the limit of the theory upon which β was calculated, and, indeed, beyond the limits of the ordinary assumption of small deflections upon which the equations are founded in all problems in which elastic curves are concerned; so that the diminution of the length is the only practicable mode of trying experiments upon small values of β. However, the values of β in actual bridges are so much larger than any we have been experimenting upon, that they belong for the most part to totally different phases of the curve,* and therefore experiments on small values are only required to test the theory.

Thus, in Godstone Bridge, the length was 30 feet. $S = 0{\cdot}19$ in.; $\beta = \frac{114395}{V^2}$; whence for velocities of 22 feet, 40 feet, 73 feet, 90 feet, we obtain $\beta = 236,\ 71{\cdot}5,\ 21{\cdot}4,\ 14{\cdot}$ respectively; of which the last belongs to a velocity, practicable indeed, but the effects of which we were not able to test.

In the Dee Bridge, $l = 98$. S varies from $\frac{7}{8}$ in. to $1\frac{1}{16}$ in.;† if we assume it equal to 1 inch, we obtain $\beta = \frac{231937}{V^2}$. In this case velocities of 20 feet, 40 feet, 70 feet, 90 feet, give values of $\beta = 580,\ 145,\ 47$, and 28 respectively.

In a bridge of 89 feet length, on the Goole line, the deflection was half an inch (*vide* Mr. Hawkshaw's evidence, Report, No.

* The principal reason of the totally different range of the values of β in the experiments, and in real bridges, respectively, is to be found in the great difference between their lengths, for as β varies (*cæteris paribus*) directly as the square of the length, and inversely as the statical deflection, it is clear that a 9-feet bar and a 30-feet bridge will at once produce a totally different set of values of β. Added to which, it is found convenient to employ a statical deflection of 1 inch or more for the sake of sufficiently developing the effects, while in real bridges the statical deflection is not greater than a quarter of an inch.

† These values of S are taken from the Report to the Commissioners of Railways, 15th June, 1847, p. 7, and consequently belong to its construction before it was strengthened.

152, &c.); this, with velocities of 25 and 90 feet, will give $\beta = 612$ and 47 respectively.

In the Ewell Bridge, $l = 48$ feet, $S = 0{\cdot}215$ in., $\beta = \frac{258789}{V^2}$, whence velocities of 25 feet and 90 feet give $\beta = 414$ and 32 respectively.

In the case of real bridges, it thus appears that β is rarely so small as 14, and may reach 600, or higher numbers, whereas, in the Portsmouth experiments, the values of β ranged between 30 and 1. In the experiments on shorter bars at Portsmouth, and in my experiments at Cambridge, still lower values of β were employed, as will presently appear. In fact, our principal experiments belong to a series of values of β that begin where those that appertain to real bridges end.*

But the better to compare the experimental results with practical cases, it will in the next place be convenient to consider the proportional increase of the central deflection of the bar that belongs to each value of β.

It has been shown in the Plate that the maximum central deflection happens when the body has reached that point of its trajectory at which the curve of the trajectory touches the corresponding curve of the bar. Every given phase of the trajectory, and therefore its appropriate value of β, has also a certain maximum central deflection in the bar, the ratio of which to the statical deflection ($= S$) can be calculated or otherwise obtained. It is not very easy to calculate it, and its value may be obtained, with sufficient accuracy for our purpose, by the drawing-board, from the curves which have been laid down from the preceding Tables, and note at foot of page 456.

However, Mr. Stokes has shown that, when β is greater than about 8, the motion of the body becomes sensibly symmetrical with

* In weak bridges still smaller values of β may be reached with high velocities. We may take, for example, the girders of the Canal Bridge near Long Eaton, which Mr. W. H. Barlow has described as exemplifying a case in which the dimensions were insufficient, and the girders removed accordingly. (Report, Minutes of Evidence, 733, and App. No. 5.) The span of the girders was 26 feet, and the statical deflection 0·3 in. This, with velocities of 70 and 90 feet, would give $\beta = 11$ and 7 respectively, and consequently increments of the statical deflection $= {\cdot}12$ and ${\cdot}2$, neglecting the inertia of the bridge, which would more than double these increments.

respect to the centre of the bridge;* and, in fact, the projections of his curves in Plate VII. show that the trajectory becoming thus nearly symmetrical, the maximum central deflection of the bar is so nearly the same as the central ordinate of the trajectory that one may be taken for the other in all cases where β is greater than 8; and of course, therefore, in real bridges, where, as we have seen, β is rarely below 14.

Now, when β is large, Mr. Stokes has given the following series,† to calculate the value of the ratio of the central deflection of the bar to S, namely (if D = central deflection of the bar):

$$\frac{D}{S} = 1 + \frac{1}{\beta} + \frac{5}{2\,\beta^2} + \frac{13}{\beta^3} +, \&c.$$

When β is equal to, or greater than 100, the first two terms of the series will be found true to the third place of decimals; therefore, substituting the value of β, we obtain $D = S + \frac{4\,V^2\,S^2}{g\,a^2}$. Hence, for a given load, the increment of the deflection due to velocity varies nearly as the square of the velocity directly, and the square of the length of the bridge inversely.

TABLE IX.—CORRESPONDING VALUES OF β AND $\frac{D}{S}$.

β	$\frac{D}{S}$	β	$\frac{D}{S}$	β	$\frac{D}{S}$
0·3	7·0	3·5	1·43	50	1·020
0·4	5·6	4·0	1·38	60	1·017
0·5	4·0	4·5	1·34	70	1·015
0·6	3·9	5	1·30	80	1·013
0·7	3·4	6	1·23	90	1·011
0·8	3·0	7	1·20	100	1·010
0·9	2·7	8	1·18	200	1·005
1·0	2·46	9	1·16	300	1·003
1·2	2·13	10	1·14	400	1·0025
1·4	1·92	12	1·12	500	1·0020
1·6	1·79	14	1·10	600	1·0017
1·8	1·72	16	1·09	700	1·0014
2·0	1·65	18	1·07	800	1·0012
2·3	1·59	20	1·06	900	1·0011
2·5	1·55	30	1·04	1000	1·0010
3·0	1·49	40	1·03	..	..

* Camb. Phil. Trans. p. 720.

† "In practical cases this series may be reduced to $1 + \frac{1}{\beta}$. The latter term is the same as would be got by taking into account the centrifugal force,

In Table IX. I have given with sufficient accuracy for our purpose, the numerical values of the ratio of the dynamical central deflection of the bar to the statical deflection, which correspond to different values of β. We see that the statical deflection is tripled when $\beta = 0{\cdot}8$, and doubled when $\beta = 1{\cdot}3$. When β becomes greater, the increment of the deflection diminishes rapidly; so that, for $\beta = 14$, it is only a tenth of the statical value, and one hundredth when $\beta = 100$. This Table explains the much greater development of the central deflection and other phenomena in the bulk of the Portsmouth experiments than in actual bridges; for by comparing Table VIII. with the three Tables relating to those experiments, at pp. 443–7, it will be seen that the great and startling increments of the deflection produced by the velocity of the load belong to small values of β (which never occur in practice), obtained by high velocities combined with the greatest loads. The values of β between 29 and 14, in these experiments, belong only to a few cases of the 15 feet velocity combined with the small deflections due to the least weights employed. And even these latter values of β are only reached in real bridges with velocities of 50 and 60 miles an hour. But the increase of deflection in these cases, as well in the Portsmouth experiments as in the above Table IX., is so small as to be of little practical importance. From Table IX., and from the values of β determined in page 484, it would appear that in real bridges, where β ranges from 600 to 14, the dynamical increment of the central statical deflection would be from ·0017 to ·1 only, whereas in the experiments, in which β ranges from 30 to 1, the same increment would acquire values from ·04 to 1·46 of the central statical deflection. It must always be remembered, however, that in our theory, the inertia of the bar or bridge has been supposed so small with respect to that of the load that it may be neglected, and consequently, as I will proceed to show, the theory, in this stage, although it serves very well to explain the general action of the forces in producing

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and substituting in the small term involving that force the radius of curvature of the equilibrium trajectory for the radius of curvature of the actual trajectory. The problem has been already considered in this manner by others by whom it has been attacked."—*Camb. Trans.* p. 724.

the effects in question, fails to account for the whole of the results obtained by experiment.

For the purpose of comparing the above-calculated values of the central deflection of the bars with the Portsmouth experiments, I will select those experiments in which the actual statical deflections were measured; for, as I have already explained, in the examination of the three first series I was compelled to calculate, upon somewhat uncertain data, the statical deflections for the purpose of obtaining the increase due to the motion of the load. But in the sixth and seventh series, the load was allowed to remain the same in each experiment, and successively increasing velocities were given to it, the statical deflection having been previously determined, and thus a cause of possible error was removed. In the seventh series, moreover, the load was made to press upon one point only of the bar, so as to remove one source of discrepancy between the theory and experiment (see page 441).

TABLE X.—PORTSMOUTH EXPERIMENTS, SIXTH AND SEVENTH SERIES.

Bars of Wrought Iron 9 ft. long, 1 in. broad, 3 in. deep.							
No. of Experiment.	Velocity, in feet per second.	Statical Deflection.	Dynamical Deflection.	Ratio of observed Deflection.	Calculated Ratio.	β	Calculated Dynamical Deflection.
Sixth Series.	15	·29	·38	1·31	1·05	27	·30
	29	·29	·50	1·72	1·19	7·24	·34
	36	·29	·62	2·14	1·34	4·7	·39
	..	·34	·53	1·56			·45
	43	·29	·46	1·59	1·46	3·3	·42
	..	·34	·47	1·38			·50
Bars of Cast Iron 4 ft. 6 in. long, 4 in. broad, 0·75 in. deep.							
Seventh Series. 1	15	·25	·48	1·92	1·17	8·7	·29
	29	..	·70	2·8	1·61	2·3	·40
	40	..	·84	3·36	2·18	1·2	·53
Bars of Cast Iron 4 ft. 6 in. long, 4 in. broad, 0·5 in. deep.							
2	15	·42	·60	1·43	1·29	5·2	·54
	29	..	1·58	3·76	1·92	1·4	·81

TABLE X.—(*continued.*)

No. of Experiment.	Velocity in feet per second.	Statical Deflection.	Dynamical Deflection.	Ratio of observed Deflection.	Calculated Ratio.	β	Calculated Dynamical Deflection.
Bars of Wrought Iron 4 ft. 6 in. long, 4 in. broad, 0·5 in. deep.							
3	15	·26	·39	1·5	1·17	8·2	·30
	29	..	·52	2·	1·61	2·2	·42
	40	..	·61	2·35	2·13	1·2	·55
	15	·34	·59	1·72	1·23	6·3	·42
	29	..	·82	2·41	1·75	1·7	·59
	40	..	1·00	2·94	2·70	·9	·92
Bars of Wrought Iron 4 ft. 6 in. long, 4 in. broad, 0·5 in. deep.							
4	15	·50	·74	1·48	1·36	4·3	·68
	40	..	1·95	3·9	3·9	·6	1·90
Bars of Steel 2 ft. 3 in. long, 2 in. broad, 0·25 in. deep.							
5	15	·35	·60	1·72	1·85	1·5	·65
	29	..	·88	2·52	5·6	·4	1·96
	44	..	1·03	2·94	..	·2	..
	15	·70	1·02	1·46	3·0	·8	2·10
	24	..	1·32	1·88	7·	·3	4·90
	29	..	1·46	2·08	..	·2	..
	34	..	1·30	1·85	..	·1	..
	44	..	1·03	1·47	..	·1	..

In Table X., after giving the observed statical and dynamical deflections, with their respective ratios, I have added three columns, containing quantities obtained by calculating in accordance with the above theory the value of β, the ratio of the dynamical to the statical deflection, and lastly the dynamical deflection.

By comparing the experimental and calculated values of the dynamical deflection it will be seen that, with the exception of the last set, the calculated values are smaller than the real values.

The excess, from its irregularity, is evidently due in part to some sources of error inseparable from the nature of the experiments, as, for example, the *set*, which shows itself by the greater difference exhibited in the case of cast iron; for the mean value of the excess in the five experiments on cast-iron bars is three-tenths (·32) of the statical deflection, whereas in the fourteen cases where wrought iron was employed, the mean value of the excess is one-tenth (·12) of the statical deflections. In the experiments on steel bars, on the other hand, the calculated deflections are greater than the actual deflections. But the values of β, in the latter case, are smaller than in the experiments on wrought and cast iron, being, with one exception, less than unity.

In the next chapter I shall show that the inertia of the bar will account for the greatest part of the discrepancies above stated between the theoretical and experimental deflections, for it will appear that it tends to increase the theoretical deflections when β is greater than about 2, and to diminish them when less. In actual bridges the jolts from the joints of the rails, and the imperfect curvature or cambering of the bridge, also tends to disturb and augment the effect, and therefore we need not be surprised to find that the increase of deflection observed in the experiments of the Commission at Ewell and Godstone Bridges was greater than the theory would have assigned, as the following Table shows:

EWELL BRIDGE.				GODSTONE BRIDGE.			
Velocity in feet per second.	β.	$\frac{D}{S}$ Computed.	$\frac{D}{S}$ Observed.	Velocity in feet per second.	β.	$\frac{D}{S}$ Computed.	$\frac{D}{S}$ Observed.
25	414	1·002	1	22	236	1·004	1·23
30	287	1·004	1·07	40	72	1·015	1·15
54	88	1·01	1·14	73	22	1·06	1·31
75	46	1·02	1·09	90	14	1·10	..
90	32	1·04	..				

In the Ewell Bridge the difference is not more than the omission of the inertia of the bridge would account for; but in the Godstone Bridge the excess is much greater than in the Ewell Bridge. The Godstone Bridge was the first upon which the experiments in question were tried, and the scaffold and registering apparatus was by no means so complete and steady as that which was used for the Ewell Bridge (figured in Plate IV.). The actual quantity to be measured (about a quarter of an inch) was so small that the least unsteadiness in the apparatus would affect its correct registration. This cause may possibly account for some part of the difference between the two experiments.

In the next place I shall proceed to show how the effect of the inertia of the bridge or bar may be examined.

CHAPTER IV.

On the Effect of the Inertia of the Bridge.

In the mathematical theory of the previous chapter it has been assumed that the mass of the bridge is so small with respect to that of the load, that its inertia may be wholly neglected. But when the trajectories obtained by the apparatus just described (figured in Plate VI.) are compared with those derived by theory under the above hypothesis, considerable differences are observed which appear due to the neglect of the inertia of the bar or bridge. For example, in Plate VII., fig. 5, I have given a series of trajectories which I obtained from my apparatus.

The bar was of steel 3 feet in length between its bearing points; its section was square and about 0·22 inch in width and depth; its weight was 8 ounces avoirdupois, and the pressure on the roller was 5 ℔s., which was very nearly the actual weight. Hence, the weight of the load was about ten times that of the bar; the central statical deflection (or S) $= 0{\cdot}764$ inch. In the figure the proportion of the ordinates to the abscissæ is greatly exaggerated (*vide* p. 471).

The values of β, which belong to the four trajectories in the figure, are respectively nearly 5, 2, 1, and ·4, as marked.

It happens that, with the exception of 5, the values of β in Mr. Stokes's Tables do not exactly coincide with the above, but it is easy to compare them with the trajectories in fig. 5, by taking the nearest cases.* Thus the curve of which $\beta = 2$ will

* It would have been better to have arranged the apparatus so as to have traced curves exactly corresponding to the values of β in Mr. Stokes's diagram (fig. 4, Plate VII.), as the change of form would thus have been more strikingly shown. But with respect to this, as well as to other parts of the investigation, I must remark that the necessity for presenting the Report of the Commission to Her Majesty before the recess, limited the time for carrying on this Inquiry, and therefore I have been compelled to leave many parts of it in an incomplete state, in order to hurry on to the conclusion. Experiments of the

lie between those which belong to 3 and $\frac{5}{4}$ in fig. 4, Plate VII., that for $\beta = 1$ a little above that which belongs to $\beta = \frac{5}{4}$, and that for $\beta = \cdot 4$ above that which belongs to $\beta = \frac{1}{2}$ or $\cdot 5$. Mr. Stokes also had foreseen * that the effect of the inertia of the bar would be to reduce the enormous deflections which occur in the second half of those theoretical trajectories which appertain to the values of β below unity. This view is fully confirmed by the experimental trajectories, of which fig. 5 contains specimens. But we will proceed to a more especial examination of the effect of the inertia of the bar.

There is a very striking similarity in the general forms of the corresponding trajectories in these two diagrams. In the curve that belongs to the smallest value of β, namely $\cdot 4$, the front of the experimental curve does not terminate so bluntly as in Mr. Stokes's diagram; and in all the trajectories it will be seen that their first intersection with the equilibrium curve takes place farther from the origin in the experimental cases than in the theoretical, which might be expected from the simplest view of the effect of inertia in the bar, which will of course retard the descent of the load at the beginning of the motion, and consequently tend to throw the first part of the trajectory higher up, and thus to carry the point of its intersection with the equilibrium curve to a greater distance from the origin of the curve.

It will be useful in this place to examine the relation between the weight of the load and the weight of the bridge in the experiments at Portsmouth, and in actual cases, in order to see what proportion the mass of the bridge bears to that of the load in reality. In the three first series of the Portsmouth experiments the weights of each cast-iron bar, 9 feet in length, were 67 ℔s., 94 ℔s., and 195 ℔s. respectively. The loads laid upon each bar in the first series varied from 560 ℔s. to 922 ℔s.; in the second series from 560 ℔s. to 1748 ℔s.; and in the third series from 560 ℔s. to 1648 ℔s. Thus the weight of the load was con-

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nature of those given above, which are intended for the elucidation of the laws of certain mechanical phenomena, do not require the minute and delicate accuracy that are essential to physical experiments, in which the most precise numerical results are to be sought for.

* Camb. Phil. Trans. p. 708.

siderably greater than that of the bridge in all these cases. The exact ratios of load to bar in the above limiting examples are, respectively, in the first series 8·3 and 13·7; in the second series 5·9 and 18·5; in the third series 2·9 and 8·4. On the whole the weight of the load is from 3 to 14 times that of the bar. In my smaller experiments, steel bars weighing from 17 ounces to 8 ounces were employed, and loads varying from 5 ℔s. to 3 ℔s.; the weight of the load was therefore from 3 to 10 times that of the bar.

In the Godstone and Ewell Bridges, upon which the Commissioners experimented, the following ratios existed. It must first be observed, that every complete railway bridge for a double line consists of two bridges, one to carry each line of rails, and that the two, although lying close together, are in reality independent structures, so that the deflection of one under the action of a passing train does not affect the other. The total weight of half the Ewell Bridge is about 30 tons, and the weight of an engine and tender nearly 40 tons, so that the load is here $\frac{1}{3}$ heavier than the bridge. In the Godstone Bridge the weight of an engine and tender was 33 tons, and of the half-bridge 25 tons, which gives nearly the same proportion as the Ewell Bridge. These may serve as examples of bridges from 50 to 30 feet span. In the Dee Bridge, of which the span is 98 feet, the half-bridge is said to weigh 90 tons, and the engine and tender 30 tons.*

The Conway tube has a clear span of 400 feet, and its weight is 1146 tons. The Britannia tube in its greatest clear span is 460 feet, and the weight of the portion that belongs to this span, namely, of 472 feet of tube, is 1400 tons.† Taking an engine and train at above 60 tons, the bridge in these two cases is more than twenty times heavier than the load.

In the experimental apparatus the weight of the load was much greater with respect to the bars than in actual bridges, partly on account of the necessity for employing very flexible bars to render

* Report of the Commissioners of Railways on the Dee Bridge, page 5. At page 3 it is stated that two engines and tenders (or 60 tons) would be at the same time on one pair of girders; this would, however, be considered as a distributed load.

† Minutes of Evidence 1232, page 359, &c. Fairbairn's Account of the Britannia and Conway Tubular Bridges, page 184.

the changes of deflection sufficiently apparent, and partly on account of the great difference of length. If bars, bearing the same ratio of weight to the load as in bridges, were tried in the apparatus, the deflections would become so small that they would be scarcely appreciable. Hence it appeared impossible to obtain trajectories corresponding to different ratios of the masses of the load and bar, which were required to teach us the effect of inertia upon the trajectory; for as it plainly appears from the above data that the mass of the bridge is too considerable to be neglected, we have next to inquire whether the inertia of the bridge increases or diminishes the amount of central deflection of the bridge, which we have calculated on the supposition of the bridge being an elastic bar without sensible inertia.

The method by which I attempted to attain this object may be thus explained. In page 453 I have stated that if an elastic bar, resting on two fixed props, be deflected by a pressure applied at any point not in the centre, it will assume the form of a certain curve, in which the greatest deflection will not be at the place where the pressure is applied, but much nearer to the centre. In fact, as the deflection is small, this curve is so nearly the same in form, whether the pressure be applied in the centre or at any other point, that we may for our present purpose assume the same equation to belong to it in all cases.

The bar may thus be considered as a system of heavy particles, so connected that if motion be given to any one of them the whole will move from their initial position, and with velocities respectively proportional to the ordinates (y) of the curve which the bar assumes. Substitute for these heavy particles a mass collected in the centre of the bar, and therefore moving with a velocity proportional to the central ordinate (Y). Then as each particle m of the bar will resist the communication of motion with a force which is as the particle itself, and the square of its velocity jointly, it can be replaced by a particle at the centre of the bar, which is equal to $\frac{m y^2}{Y^2}$; and hence if this central mass be equal to the sum of these $\frac{m y^2}{Y^2}$, the effect of its inertia will be the same as that of the whole of the particles of the bar. Calculating this sum from the equation to the curve we find it to represent 0·486 of the mass of the bar, or one-half nearly. It thus appears that in considering the effect of the inertia of the bar, we may suppose a mass equal to one-half of its weight to be collected at the centre.

In the next place let there be a rod, $p\,q\,r$, below the bar (fig. 4, Plate VI.), balanced upon knife edges at q, and provided with a sliding weight at each end, and suppose these weights and the rod to be adjusted in equilibrium about the centre of motion; let k be the radius of gyration of the system, Mk^2 its moment of inertia, and r the radial distance of the point p from the centre, then this system will resist the communication of motion to the point p, with a force equal to that of a mass $\frac{Mk^2}{r^2}$ collected at that point.

If the point p be connected to the centre o of the trial bar by a light link rod, this point will move with the same velocity as the centre of the trial bar, whenever motion is communicated to any point of the bar, and consequently the balance and its weights will revolve about the centre q. The effect of this arrangement, therefore, is the same as if a mass $\frac{Mk^2}{r^2}$ were collected in the centre of the bar. By altering the distance of the weights from the centre, always keeping them in equilibrium, we can increase or diminish the value of $\frac{Mk^2}{r^2}$ at pleasure, and as the system is in equilibrium we do not thereby affect the deflections of the bar. Thus we have at our disposal an artificial inertia applicable to the bar, by means of which we can, retaining the same bar and the same load, try successive experiments, and obtain successive trajectories appertaining to various proportions between the inertia of the load and that of the bar. Half the weight of the bar must of course be added to the mass $\frac{Mk^2}{r^2}$, which represents the inertia added by the 'Inertial Balance.' *

The link op was formed of flat steel, and was connected to the bar by a contrivance shown at large in fig. 8. The upper half of

* It may be necessary to remind my reader that the whole of this investigation proceeds upon the supposition that the deflections of the bar communicated by the travelling load take place simultaneously throughout its

the link was divided into two branches, and bent into the form shown in the drawing. Each branch carried a steel centre-point, and the branches could be set at any required distance by the thumb-screw and nut; their elasticity of course pressing them outwards. Two centre punch-holes were made in the sides of the bar at its middle point, and the steel points of the branches were adjusted so as to allow those points to enter the punched holes, and play therein with the least possible shake and friction. The lower end of the link is pierced and enters a slit in a small steel arm at p, screwed to the end of the lever of the balance. A wire pin passing through the holes drilled in the arm and link forms the lower joint; the lever of the balance is a square bar of oak, and graduated in ounces avoirdupois, so that the weights set to any given number of ounces on the scale, and of course balanced, shall represent the equivalent mass added to the half-weight of the bar; the sliding weights were $3\frac{1}{4}$ lbs. each.

From several sets of curves drawn with this apparatus, I have selected the three groups in figs. 6, 7, 8, Plate VIII. These trajectories were all obtained from a steel bar 4 feet long, of a square section 0·23 inch broad on each side. Its weight was 11 ounces avoirdupois, and the carriage was loaded with weights that gave an effective pressure of 3 lbs. All the curves in each group were drawn with the same velocity, and consequently have the same value of β. The curves in fig. 6 were drawn with a velocity of 7·7 feet per second, and $\beta = 6$. The curves in fig. 7 with a velocity of 11·9 feet, and $\beta = 2{\cdot}4$. The curves in fig. 8 with a velocity of 16·6 feet, and $\beta = 1{\cdot}2$.

The differences between the curves in each group are due to the different proportions of inertia introduced by the 'Inertial Balance.' To distinguish the several curves in each group from each other, different kinds of lines are employed. The equilibrium trajectory is necessarily the same in all the groups. This is distinguished by a plain thick line, and moreover has the name

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length, and that consequently the bar at every instant of the passage of the load is bent into the same curve which it would assume if the point of application of the load were pressed down statically to the same position. See p. 456.

written upon it. The interrupted dots in cloudy masses indicate the course of the curve that corresponds to β in Mr. Stokes's Tables, and therefore to the case in which the inertia of the bar or bridge is so small as to be wholly neglected. The plain continuous line marked B, which lies close to it in the three groups, is the trajectory obtained from the bar before the Inertial Balance was connected to it; and therefore the ratio of the mass of the load to the bar in this case is more than 4 to 1.

The next trajectory in order is a dotted line, which was obtained by so adjusting the balance, that its effect should be to make the masses of the load and bar equal. The next, an interrupted line, similarly belongs to the case in which the mass of the bar is double that of the load; and the last, an interrupted line with longer strokes, alternating with dots, represents the case in which the mass of the bar is triple that of the load. Now it will be seen, on examining the three groups in figs. 6, 7, and 8, that the five curves do not follow throughout in the same order in all of them.

In the first part of the curves, indeed, as they start from their origin at the beginning of the bar, the order in all is the same; the increase of inertia uniformly throws the trajectory higher up, and we always find the equilibrium curve the lowest; the theoretical curve, in which the bar has no appreciable inertia, the next above, and the others rising in the order of their increased inertia.

All the dynamical curves intersect the equilibrium trajectory, and they all sag below it more or less in the second half. The increasing inertia carries the intersection of the curves with the equilibrium trajectory farther from the origin in every instance, and all the intersections lie farther from the origin in fig. 7 than in fig. 6, and still farther in fig. 8; that is to say, farther in the smaller values of β than in the larger.

But the effect upon the amount of the maximum depression of the trajectory is different in each of the three values of β. In fig. 6, the increase of inertia causes the successive trajectories to fall lower and lower, and thus to occasion a *greater central deflection* of the bar, as the mass of the bar is increased with respect to that of the load. In fig. 8, the increase of the inertia, on the contrary, diminishes the maximum deflection of the successive trajectories, and thus occasions a *less central deflection* of the bar as the mass of the bar is increased.

To show this more clearly, I have introduced dotted lines in figs. 6 and 8, marked $T \ldots . T$, $B \ldots . B$, $1 \ldots . 1$, $2 \ldots . 2$, $3 \ldots . 3$, which lines will be observed each to begin from a point on the central ordinate of the curve, and to end in contact with the trajectory in order.* Each dotted line represents the part of the bar which lies between its centre and the trajectory, at the moment of greatest depression; and therefore shows the greatest central deflection of the bar that corresponds to each trajectory. It thus appears that in fig. 6 the increase of inertia in the bar carries it lower and lower, and in fig. 8 the reverse happens; and of course for smaller values of β the latter effect would be more strikingly developed; because, as we have seen, the central deflection goes on increasing as β diminishes, when the inertia of the bar is wholly neglected.

The succession of the curves shows pretty clearly, however, that if still more and more inertia were given to the bar in fig. 6, the series of trajectories would reach a maximum depression and then begin to rise; after which a further increase of inertia would diminish the central deflections, as in the curves of fig. 8. And this effect is shown in fig. 7, where the maximum deflection or sag of the trajectory which belongs to the bar alone sinks a little below that of the theoretical trajectory or curve of no inertia; and the next curve, in which the masses of bar and load are equal, sinks still a little lower. But the following curves, which belong to a double and triple ratio of these masses, rise higher and higher, and the central deflections of the bar follow in the same order.

It would seem from this, that for any given ratio of the masses of the bar and load some value of β may be found, for which a small variation in the ratio would neither increase nor diminish the central deflection of the bar; while for smaller values of β, the increase of inertia in the bar would diminish the central deflection, and for greater values of β, the reverse. It would require a long series of experiments to determine these values with accuracy,

* In these figures T denotes the theoretical trajectory, B the bar alone, in which the ratio of the mass of the bar to that of the load is $\frac{1}{2}$, and the figures 1, 2, 3, denote ratios of their respective values.

which the short time assigned for this research has made it impossible for me to attempt; but they may be roughly estimated as follows:

In fig. 7, $\beta = 2{\cdot}4$, and the trajectory which sinks the lowest in this figure is that which corresponds to the ratio of equality between the bar and load. It is evident from the manner in which the deflections of the bar succeed each other that the greatest deflection for this value of β would lie a little below that marked 1 in the figure, and probably correspond to about $\frac{B}{L} = {\cdot}7$ (where B is the mass of the bar and L of the load). Hence, to bring the trajectory for $\frac{B}{L} = 1$ to its maximum bar-deflection, a little larger value of β must be taken; and probably $\beta = 3$ will be very nearly the value that corresponds to the exact position of the maximum depression belonging to this trajectory.

In fig. 6, where $\beta = 6$, the last trajectory that the proportions of my apparatus enabled me to obtain belongs to the triple ratio of the masses. It seems probable that if two or perhaps three more had been drawn to correspond to the succeeding ratios, the maximum deflection would have been reached for this value of β; and that therefore the trajectory corresponding to the ratio $\frac{B}{L} = 6$ will be very nearly the one sought for.

Now Mr. Stokes has shown, as we shall see below, that when β is moderately large, and the above ratio also large, the trajectory remains constant if β varies as the ratio, that is to say, $\frac{B}{L} = c\beta$, where c is a constant. As we have just found a case in which the maximum deflection is given by a trajectory, in which the ratio of the weights of bar to load is nearly *equal* to β, and β is moderately large, we shall not err much in taking $c = 1$, and therefore in saying that the maximum deflection for any given large value of β will happen when the mass of the bridge is nearly β times that of the load.* This is sufficient to show us

* Subsequent researches of Mr. Stokes showed that in moderately large values of β, and large values of the ratio $\frac{B}{L}$, we have for the maximum deflection $\frac{B}{L\beta} = {\cdot}823$, which differs from unity by $\cdot177$ only. (See note, p. 504.)

that in all practical cases the inertia of the bridge will increase the deflection which is due to the velocity of the load; for in practice the value of β is always much greater than the ratio of the weights of the bridge and load. But to return to fig. 8; in this figure the central deflection (T) of the bar produced by the theoretical trajectory very nearly coincides with (B) that which is due to the trajectory for the bar alone in which $\frac{B}{L} = \frac{1}{4}$; more closely, in fact, than the figure shows, in which the distance between these two curves is slightly exaggerated. In this group, therefore, it happens that we have nearly the value of β, for which the maximum deflection of the bar is due to the theoretical curve. This value of β is 1·2, or unity, very nearly.

The general results of the experiments with the inertial balance may be therefore stated as follows:

(1.) For all values of β less than about unity, the least sensible inertia added to the bar will diminish the central deflection due to the theoretical trajectory, namely, that in which the bar is supposed to have no inertia.

(2.) For all values of β greater than about unity, inertia gradually added to the bar will at first increase the central deflection due to the theoretical trajectory, will then bring it to a maximum, and finally will diminish it.

(3.) The ratio of the masses of the bar to the load that corresponds to this maximum effect will be very nearly unity for $\beta = 3$, and for larger values of β and of $\frac{B}{L}$ will be expressed by the equation $B = \beta L$ (or more accurately $B = {\cdot}823\, \beta L$).

The differences between the theoretical trajectories of fig. 4, Plate VII., and the experimental trajectories of fig. 5, are now explained. When the inertia of the bar is neglected, it was shown that for small values of β, the deflections of the bar became excessively great, and that when β is less than $\frac{1}{4}$, the tangent at the end of the trajectory is vertical, and the central deflection of the bar and the tendency to break the bridge become infinite. Mr. Stokes had already explained these startling results, by supposing that the inertia of the bridge was the cause of the practical modifications of these consequences; but without experiment it was impossible to ascertain that the inertia would, in cases where β was greater than unity, produce the opposite effect of increasing the deflections, or indeed to understand the exact nature of the influence which different proportions between the inertia of the bar and load would have upon the trajectories.

In the last chapter it was shown that in real bridges β is rarely so small as 14, and hence it follows from the experiments of the inertial balance that the inertia of a bridge will tend to increase the deflections due to the theoretical trajectory of no inertia, which have been exhibited in Table IX. (p. 486). And the result is perfectly in conformity with the analysis of the sixth and seventh series of the Portsmouth experiments, given in Table X. (p. 488), in which the deflections for values of β greater than unity were all greater by about one-tenth, more or less, than the theoretical deflections. A similar increase was obtained from the experiments on the Godstone and Ewell Bridges, which has been now shown to be due, in part at least, to the inertia of the bridge. It also appeared from the same seventh series (Table X.) that when β was less than unity, the experimental deflections of the bar were less than the theoretical deflections of Table IX., which is also in accordance with the results obtained from the inertial balance.

It becomes therefore a point of great interest to determine the exact increment of the deflection of a real bridge that would be due to its inertia. My experiments, besides being limited to values of β considerably below 14, and therefore smaller than those that belong to practice, were, from want of time, too few in number and deficient in precision to give accurate numerical results, although amply exact enough to show the laws of the phenomena. The following values of the deflections in fig. 6 are probably not far from the truth, although subsequent and repeated experiments would be required to correct them.

In this figure $\beta = 6$, and the ratios of the dynamical to the statical deflection $\left(\frac{D}{S}\right)$ corresponding to the different ratios of inertia $\left(\frac{B}{L}\right)$ are given in the following Table:

$\frac{B}{L}$	0	$\frac{1}{4}$	1	2	3
$\frac{D}{S}$	1·23	1·3	1·52	1·67	1·78

Thus for this value of β, the theoretical deflection with no inertia is increased by about ·07 when the bar has a mass of one-fourth of the load, and by ·3 when the masses of the two are equal.

These results have been obtained from experiments made on a small scale, but by setting down the equations that relate to the problem in its general form, Mr. Stokes succeeds in showing that if we have two systems in which the ratio of L to B is the same, and we conceive the travelling weights to move over the two bridges respectively, with velocities ranging from 0 to ∞, the trajectories described in the one case and the deflections of the bridge correspond exactly to the trajectories and deflections in the other case; so that to pass from the one to the other, it will be sufficient to alter all horizontal lines on the same scale as the length of the bridge, and all vertical lines on the same scale as the central statical deflection. The velocity in the one, which corresponds to a given velocity in the other, is determined by the value of the constant β.* We are thus furnished with the important result, that if by experiment a certain form of the trajectory be obtained, the same form will belong to every case in which the ratio of the masses of the bar and load is the same as in the experiment, and also the value of β the same.

Thus, by the use of the inertial balance, we shall be able to construct with facility a *dynamical model* of a large system, which we may wish to investigate experimentally. To take a numerical example, let there be a load of 25 tons moving over a girder bridge 40 feet long and weighing 25 tons, the central statical deflection being $\frac{2}{3}$ inch, and the velocity of the load 30 miles an hour, or 44 feet per second (this will give $\beta = 24$). Suppose the trial bar in the model to be 4 feet long, and the central statical deflection due to the travelling weight to be $1\frac{1}{2}$ inch. We must, in the first place, adjust the inertial balance so as to give to the bar a distributed mass equal to the mass of the load. We must now give to the carriage such a velocity as shall render β the same in the two cases. Since β is constant when the velocity varies directly as the length of the bridge, and inversely as the square root of the central statical deflection, we must alter the velocity in the direct ratio of 40 to 4, or 10 to 1, and in the

* Camb. Phil. Trans. p. 727.

inverse ratio of $\sqrt{\frac{2}{3}}$ to $\sqrt{\frac{3}{2}}$, or 2 to 3. Hence the velocity required in the model is $44 \times \frac{1}{10} \times \frac{2}{3}$, or 2·9 feet per second.

But to determine experimentally the amounts for high values of β, an apparatus calculated to operate upon longer bars with much less velocity would be necessary; fortunately, however, it happens that the investigations of Mr. Stokes will assist us in obtaining, at least in part, the information we require.

During the progress of my experiments above related, this gentleman had been simultaneously carrying on his theoretical researches with a view of determining the effect of the inertia of the bridge, which in the previous investigation had been neglected; and although he did not succeed in obtaining the complete solution of this most intricate problem, he rendered the greatest service to the question by obtaining an approximate solution; namely, one limited by the following condition, that the value of β be large or moderately large, and that the mass of the travelling body be *small* compared with the mass of the bridge.

Small values of β never occur with real bridges, and therefore the first condition includes all practical cases. Unfortunately the mass of the travelling body in practice is very nearly equal to that of the bridge, so that the latter condition does not represent the practical cases so well. But Mr. Stokes, by giving in the first place a solution of the case in which the mass of the bridge is neglected, and in the next place one in which the mass of the load is neglected, or its effect reduced to a travelling *pressure*, has solved the problem in the two extreme cases between which the practical examples lie; and has thus enabled us, assisted by the experiments, to calculate with sufficient accuracy the amount of additional deflection which is due to the velocity of the travelling load. I shall proceed, therefore, to explain the results of this most valuable addition to Mr. Stokes's former investigation, as nearly as possible in his own words, referring, as before, for the analysis to the original Paper in the 'Cambridge Philosophical Transactions.'

The general equations (which are given in the original paper) proved too complex to be manageable, but by introducing the limiting conditions above mentioned, namely that β be large or moderately large, and that the mass of the travelling body be small compared with the mass of the bridge, Mr. Stokes succeeded in reducing the equations to a form which admitted of a complete solution, and hence has calculated the ordinates of the trajectories in a sufficient number of cases; so as to enable us to lay down the curves, and thus to understand the nature of the motion.

It appears that in these trajectories each phase is characterized by the value of a certain constant quantity, q, which occupies in this part of the investigation a similar office to the β of the previous pages.

This quantity, q, is defined as follows: let S be the central statical deflection, M the mass of the travelling body, M^1 the mass of the bar or bridge. Then

$$q^2 = \frac{63}{31}\frac{Mg}{M^1 V^2 S} = \frac{1008}{31}\frac{M\beta}{M^1}.*$$

* From this expression, it appears that if β vary directly as $\frac{M^1}{M}$ the value of q, and therefore the form of the trajectory remains unaltered; whence, having obtained from my experiments that, when $\beta = 6$, the trajectory which corresponds to the maximum deflection of the bar is very nearly that which belongs to the ratio $\frac{M^1}{M} = 6$, I inferred that we may roughly take $\frac{M_1}{M} = \beta$ to represent the case of the maximum deflection. Probably neither the ratio of the masses nor the value of β in this case is large enough to satisfy the conditions, upon which the above expression is founded, with sufficient accuracy. Upon this, however, Mr. Stokes has kindly furnished me with the following note: "In fig. C, it appears that the maximum curve of deflection lies between 3 and 4 (that is, between those which correspond to $\frac{2q}{\pi} = 3$ and 4). I have found by interpolation,

$\frac{2q}{\pi}$	Maximum value of $\frac{D}{S}$.
3	1·717
4	1·697
5	1·580

And again, by interpolation, the maximum value of $\frac{D}{S} = 1{\cdot}721$, in which case $\frac{M_1}{M\beta} = {\cdot}823$, which differs only by ·177 from the result to which you were led by experiment."

Conceive the travelling mass M removed, and suppose the bar depressed through a small space and then left to itself to oscillate. It can be shown that if P be the period of motion, or twice the time of oscillation from rest to rest, S, the central statical deflection produced by a mass equal to that of the bridge and expressed in inches, and τ the time in seconds that the body takes to travel over the bridge, we have

$$P = 2\pi \sqrt{\frac{31}{63}\frac{S_1}{g}}; \quad q = 2\pi \frac{\tau}{P}.*$$

Hence the numbers 1, 2, 3, &c., written at the head of Tables A and B, and against the curves in Plate IX., represent the number of quarter periods of oscillation of the bridge which elapse during the passage of the body over it. This consideration will materially assist us in understanding the nature of the motion. It should be remarked, too, that q is increased by diminishing either the velocity of the body or the inertia of the bridge.

In Table A, the length of the bar is supposed to be divided into 20 equal parts for abscissæ, and the values of the ordinates $\frac{y}{S}$, corresponding to each of the 20 values of x, are given in the Table for 11 values of $\frac{2q}{\pi}$. The curves of this Table are the trajectories of the moving body, similarly with the trajectories of Plates VII. and VIII. To prevent the confusion which would have arisen if all these trajectories had been laid down in one figure, as in Plate VII., they have been divided into two groups in Plate IX. Fig. *B* contains those which appertain to the quarter periods 1, 2, 3, 4, 5, 6, and fig. *D* those which belong to the quarter periods 8, 10, and 16, 12 being omitted to prevent confusion. In each of these figures the equilibrium trajectory is laid down as a standard by which to compare them with each other, and with the trajectories already given.

Table B, however, to which correspond figs. *C* and *E* in Plate IX., refers to a different kind of curve, which may be termed the deflection curve. It is headed 'Values of $\frac{D}{S}$,' D

* If we suppose τ expressed in seconds, and S_1 in inches, we must put $g = 32{\cdot}2 \times 12 = 386$, nearly, and we get $q = \frac{28 \cdot \tau}{\sqrt{S_1}}$ . . . . (69).—*Camb. Phil. Trans.* p. 732.

being, as already explained, the central deflection of the bar which corresponds to any value of y.

The ordinate in these curves, therefore, represents the central deflection of the bar (expressed in its relation to S as those of the trajectories are), when the moving body has travelled over a distance represented by the abscissa, and hence the entire curve delineates the vertical motion of the centre of the bar during the progression of the body from one end to the other of the bar. It is, in fact, the curve which would be delineated by a pencil fixed to the centre of the bar (as in the apparatus described in the first chapter of this Essay), tracing its line upon a board that travels horizontally. If this board travelled uniformly at a rate equal to that of the body, the length of this curve would be exactly the same as that of the trajectory. This, for convenience sake, has been made the case with the figures in Plate IX.; for thus each of these deflection curves in figs. *C* and *E* lies immediately below the trajectory which belongs to it in figs. *B* and *D* respectively; in such a manner that when the body is at any given point in one of these trajectories, the magnitude of the central deflection of the bar at that instant is to be found in the ordinate of the deflection curve which is vertically beneath it.

TABLE A.

x	Values of $\frac{y}{S}$ when $\frac{2q}{\pi}$ is equal to										
	1	2	3	4	5	6	8	10	12	16	∞
·00	·000	·000	·000	·000	·000	·000	·000	·000	·000	·000	·000
·05	·001	·001	·001	·001	·001	·001	·002	·003	·004	·006	·025
·10	·003	·004	·007	·008	·012	·017	·025	·037	·050	·075	·096
·15	·008	·013	·022	·034	·050	·067	·108	·150	·190	·244	·207
·20	·015	·031	·059	·095	·137	·184	·279	·360	·414	·420	·344
·25	·029	·056	·126	·203	·290	·378	·532	·621	·630	·504	·496
·30	·045	·117	·230	·366	·509	·640	·814	·839	·744	·560	·646
·35	·063	·191	·374	·581	·778	·934	1·054	·940	·755	·727	·780
·40	·096	·285	·550	·828	1·062	1·205	1·178	·921	·759	·969	·886
·45	·133	·394	·748	1·085	1·316	1·395	1·164	·849	·846	1·084	·954
·50	·169	·516	·947	1·310	1·492	1·460	1·036	·812	1·004	·991	·977
·55	·210	·632	1·126	1·473	1·555	1·387	·860	·850	1·114	·852	·954
·60	·244	·739	1·258	1·542	1·487	1·191	·704	·923	1·062	·830	·886
·65	·274	·816	1·325	1·502	1·300	·917	·609	·942	·848	·857	·780
·70	·292	·854	1·308	1·352	1·022	·626	·565	·839	·584	·752	·646
·75	·298	·842	1·205	1·111	·705	·369	·532	·619	·391	·488	·496
·80	·282	·770	1·020	·814	·402	·180	·462	·359	·297	·280	·344
·85	·245	·644	·774	·509	·161	·069	·337	·149	·237	·178	·207
·90	·184	·463	·498	·244	·012	·020	·182	·037	·150	·121	·096
·95	·103	·243	·224	·064	−·037	·004	·051	·003	·047	·044	·025
1·00	·000	·000	·000	·000	·000	·000	·000	·000	·000	·000	·000

TABLE B.

x	Values of $\frac{D}{S}$ when $\frac{2q}{\pi}$ is equal to										
	1	2	3	4	5	6	8	10	12	16	∞
·00	·000	·000	·000	·000	·000	·000	·000	·000	·000	·000	·000
·05	·004	·004	·005	·006	·007	·008	·014	·019	·025	·041	·156
·10	·009	·013	·022	·027	·037	·053	·081	·117	·158	·239	·307
·15	·017	·028	·048	·075	·108	·146	·234	·327	·412	·530	·449
·20	·025	·052	·099	·159	·231	·309	·469	·607	·696	·707	·580
·25	·041	·093	·177	·285	·406	·531	·746	·871	·884	·707	·696
·30	·056	·144	·282	·451	·626	·787	1·003	1·031	·915	·689	·794
·35	·070	·214	·418	·650	·871	1·045	1·180	1·052	·845	·814	·873
·40	·100	·300	·578	·870	1·115	1·265	1·238	·967	·796	1·017	·930
·45	·134	·399	·757	1·097	1·332	1·412	1·178	·859	·856	1·097	·965
·50	·169	·516	·947	1·310	1·492	1·460	1·036	·812	1·004	·991	·977
·55	·213	·640	1·139	1·491	1·574	1·403	·870	·860	1·127	·862	·965
·60	·256	·776	1·321	1·619	1·562	1·250	·739	·969	1·115	·872	·930
·65	·306	·913	1·482	1·681	1·454	1·027	·682	1·054	·948	·959	·873
·70	·359	1·050	1·609	1·663	1·257	·769	·695	1·031	·718	·924	·794
·75	·419	1·181	1·691	1·560	·990	·517	·746	·869	·549	·707	·696
·80	·475	1·296	1·717	1·371	·677	·303	·777	·604	·499	·472	·580
·85	·533	1·399	1·681	1·106	·350	·149	·733	·325	·516	·384	·449
·90	·586	1·476	1·588	·776	·037	·064	·579	·117	·477	·385	·307
·95	·646	1·525	1·402	·400	−·234	·025	·321	·021	·296	·276	·156
1·00	·699	1·540	1·158	·000	−·446	·019	·000	·001	−·001	·000	·000

"In the trajectory 1, fig. *B*, the ordinates are small, because the body passed over before there was time to produce much deflection in the bridge; at least, except towards the end of the body's course, where even a large deflection of the bridge would produce only a small deflection of the body. The corresponding deflection curve (curve 1, fig. *C*) shows that the bridge was depressed, and that its deflection was rapidly increasing when the body left it.

"When the body is made to move with velocities successively one-half and one-third of the former velocity, more time is allowed for deflecting the bridge, and the trajectories marked 2, 3, are described, in which the ordinates are far larger than in that marked 1. The deflections, too, as appears from fig. *C*, are much larger than before, or at least much larger than any deflection which was produced in the first case while the body remained on the bridge. It appears from Table B, or from fig. *C*, that the greatest deflection occurs in the case of the third curve nearly, and that it exceeds the central statical deflection by about three-fourths of the whole.

"When the velocity is considerably diminished, the bridge has time to make several oscillations while the body is going over it. These oscillations may be easily observed in figs. *C* and *E*, more especially in the latter; and their effect on the form of the trajectory, which may indeed be readily understood from fig. *C*, will be seen on referring to figs. *B* and *D*." . . . . . .*

"When q is large,† as is the case in practice, the following expression will give with sufficient accuracy the value of the central deflection D_1.

$$\frac{D_1}{S} = -\frac{25}{8 \cdot q} \sin q x.$$

So that the central deflection is liable to be alternately increased

* Camb. Phil. Trans. page 733.

† Camb. Phil. Trans. p. 732 and 733. "As every thing depends on the value of q, in the approximate investigation in which the inertia of the bridge is taken into account, it will be proper to consider farther the meaning of this constant. In the first place it is to be observed that, although M appears in the equation $q^2 = \frac{1008\, M \beta}{31\, M^1}$, q is really independent of the mass of the travelling body; for when M alone varies, β varies inversely as S, and S

and decreased by the fraction $\frac{25}{8 \cdot q}$ of the central statical deflection. And it can also be shown that

$$\frac{25}{8}\frac{1}{q} = \cdot 55 \sqrt{\frac{M^1}{M\beta}} = \cdot 112 \frac{\sqrt{S_1}}{\tau}.$$

It is to be remembered that, in the latter of these expressions, the units of space and time are an inch and a second respectively. Since the difference between the pressure on the bridge and the weight of the body is neglected in the investigation in which the inertia of the bridge is considered, it is evident that the result will be sensibly the same, whether the bridge in its natural position be straight, or be slightly raised towards the centre, or, as it is technically called, *cambered*. The increase of deflection in the case first investigated would be diminished by a camber.

"In this Paper the problem has been worked out, or worked out approximately, only in the two extreme cases in which the mass of the travelling body is infinitely great and infinitely small respectively, compared with the mass of the bridge. The causes of the increase of deflection in these two extreme cases are quite distinct. In the former case the increase of deflection depends entirely on the difference between the pressure on the bridge and the weight of the body, and may be regarded as depending on the centrifugal force. In the latter, the effect depends on the manner in which the force, regarded as a function of the time, is applied to the bridge. In practical cases the masses of the body and of the bridge are generally comparable with each other, and the two effects are mixed up in the actual result. Nevertheless if we find that each effect, taken separately, is insensible, or so small as to be of no practical importance, we may conclude, without much

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varies directly as M, so that q remains constant. To get rid of the apparent dependence of q on M, let S_1 be the central statical deflection produced by a mass equal to that of the bridge, and at the same time restore the general unit of length. If x continue to denote the ratio of the abscissa of the body to the length of the bridge, q will be numerical, and therefore, to restore the general unit of length, it will be sufficient to take the general expression for β, namely, $\beta = \frac{g\,a}{4\,V^2\,S}$; let moreover τ be the time the body takes to travel over the bridge, $\therefore\ 2\,a = V\tau$, and we get $q^2 = \frac{63}{31} \cdot \frac{g\,\tau^2}{S_1}$."

fear of error, that the actual effect is insignificant. Now we have seen that if we take only the most important terms, the increase of deflection is measured by the fractions $\frac{1}{\beta}$ (page 486 above) and $\frac{25}{8 \, . \, q}$ of S. It is only when these fractions are both small that we are at liberty to neglect all but the most important terms; but in practical cases they are actually small. The magnitude of these fractions will enable us to judge of the amount of the actual effect.

"To take a numerical example, lying within practical limits, let the span of a girder bridge be 44 feet, and suppose a weight equal to $\frac{4}{3}$ of the weight of the bridge to cause a deflection of $\frac{1}{5}$ inch. These are nearly the circumstances of the Ewell Bridge, mentioned in the Report of the Commissioners.

"In this case $S_1 = \frac{3}{4} \times \cdot 2 = \cdot 15$; and if the velocity be 44 feet in a second, or 30 miles an hour, we have $\tau = 1$, and therefore from the second of the formulæ just stated,

$$\frac{25}{8 \, . \, q} = \cdot 0434 \qquad q = 72 \cdot 1 = 45 \cdot 9 \times \frac{\pi}{4}.$$

The travelling load being supposed to produce a deflection of $\cdot 2$ inch, we have

$$\beta = 127 \therefore \frac{1}{\beta} = \cdot 0079.$$

Hence in this case the increase of the deflection due to the inertia of the bridge is between five and six times as great as that obtained by considering the bridge as infinitely light; but in neither case is the deflection important. With a velocity of 60 miles an hour, the increase of deflection $\cdot 0434 \, S$ would be doubled.

"In the case of one of the long tubes of the Britannia Bridge, β must be extremely large; but on account of the enormous mass of the tube, it might be feared that the effect of the inertia of the tube itself would be of importance. To make a supposition every way disadvantageous, regard the tube as unconnected with the rest of the structure, and suppose the weight of the whole train collected at one point. The clear span of one of the great tubes is 460 feet, and the weight of the tube 1400 tons.

"When the platform on which the tube had been built was removed, the centre sunk 10 inches, which was very nearly what had been calculated, so that the bottom became very nearly straight, since, in anticipation of the deflection which would be produced by the weight of the tube itself, it had been originally built curved upwards. Since a uniformly distributed weight produces the same deflection as $\frac{5}{8}$ of the same weight placed at the centre, we have in this case $S_1 = \frac{8}{5} \times 10 = 16$; and supposing the train to be going at the rate of 30 miles an hour, we have $\tau = \frac{460}{44} = 10{\cdot}5$, nearly. Hence in this case $\frac{25}{8{\cdot}q} = {\cdot}043$, or $\frac{1}{23}$, nearly; so that the increase of deflection due to the inertia of the bridge is unimportant."*

It appears from the above that the increase of deflection is

* In the course of the investigations undertaken by Mr. Stokes and myself, our attention was directed to an able Paper by Mr. Cox, 'On the Dynamical Deflection and Strain of Railway Girders,' which is printed in the Civil Engineers' and Architects' Journal for September, 1848. This Paper is purely theoretical, that is to say, that although the results are applied to practical cases, it is not founded upon experiments; and consequently the subject is looked at in a totally different light from that under which we have viewed it. The author has employed methods of approximation which, although they have not apparently vitiated his results, as far as real bridges are concerned, would yet cause them to fail utterly if applied to the interpretation of experiments, such as those contained in the present Essay. This must be carefully borne in mind in considering the Paper in question, which will well repay perusal. The reasons for this failure are explained in the following extract from Mr. Stokes's Paper in the Cambridge Philosophical Transactions (page 725):—"In this article the subject is treated in a very original and striking manner. There is, however, one conclusion at which Mr. Cox has arrived, which is so directly opposed to the conclusions to which I have been led, that I feel compelled to notice it. By reasoning founded on the principle of *vis viva*, Mr. Cox has arrived at the result that the moving body cannot in any case produce a deflection greater than double the central statical deflection, the elasticity of the bridge being supposed perfect. But among the sources of labouring force which can be employed in deflecting the bridge, Mr. Cox has omitted to consider the *vis viva* arising from the horizontal motion of the body. It is possible to conceive beforehand that a portion of this *vis viva* should be converted into labouring force, which is expended in deflecting the bridge; and this is, in fact, precisely what takes place. During the first part of the motion, the horizontal component of the reaction of the bridge against

measured by the two fractions $\frac{1}{\beta}$ and $\frac{25}{8.q}$ of S respectively in the two extreme cases in which the mass of the bridge or the mass of the body is neglected; and that, in practice, where these masses are very nearly equal, their effects are mixed up together in a manner that remains to be developed from the theoretical equation. It is extremely desirable, however, that we should in the mean time obtain some estimate of the practical effect of the inertia of the bridge. This Mr. Stokes suggested to me might

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the body impels the body forwards, and therefore increases the *vis viva* due to the horizontal motion; and the labouring force which produces this increase being derived from the bridge, the bridge is less deflected than it would have been had the horizontal velocity of the body been unchanged. But during the latter part of the motion the horizontal component of the reaction acts backwards, and a portion of the *vis viva* due to the horizontal motion of the body is continually converted into labouring force, which is stored up in the bridge. Now, on account of the asymmetry of the motion, the direction of the reaction is more inclined to the vertical when the body is moving over the second half of the bridge than when it is moving over the first half, and moreover the reaction itself is greater, and therefore, on both accounts, more *vis viva* depending upon the horizontal motion is destroyed in the latter portion of the body's course than is generated in the former portion: and therefore, on the whole, the bridge is more deflected than it would have been had the horizontal velocity of the body remained unchanged.

"It is true that the change of horizontal velocity is small; but nevertheless, in this mode of treating the subject, it must be taken into account; for, in applying to the problem the principle of *vis viva*, we are concerned with the square of the vertical velocity, and we must not omit any quantities which are comparable with that square. Now the square of the absolute velocity of the body is equal to the sum of the squares of the horizontal and vertical velocities, and the change in the square of the horizontal velocity depends upon the product of the horizontal velocity and the change of horizontal velocity; but this product is not small in comparison with the square of the vertical velocity."

I have great pleasure in taking this opportunity of expressing my acknowledgments to my excellent friend and fellow-labourer, Professor Stokes, for his kind and friendly co-operation with me in these investigations. I must also regret that the abstruse nature of his portion of them has prevented me from giving them at length, and thereby compelled me to do him great injustice by presenting his results only, apart from the admirable reasoning, by means of which they were obtained. It may be well to mention, however, that this course was adopted with his entire concurrence.

be roughly and empirically done by supposing the two fractions in question to represent the separate effects of the inertia of the bridge and load, and taking their sum to represent the total effect. Upon calculating the increments of the statical deflection in this manner, that were obtained experimentally by the inertial balance, (and given in the Table in page 501 above,) and comparing the results, it appears that the agreement is sufficiently close, as the following Table will show.

—	Values of $\frac{B}{L}$.			
	$\frac{1}{4}$	1	2	3
Experimental increments, $\beta = 6$ . . .	·3	·52	·67	·78
Calculated increments $\beta = 5$ . . .	·42	·55	·65	·72
Calculated increments $\beta = 6$ . . .	·34	·45	·54	·62

For larger values of β, in which the increments are smaller, we may suppose the errors to be less sensible, and therefore I have calculated the following Table for several values of β, and on the supposition that the masses of the bridge and load are equal, and, therefore, $\frac{25}{8 \cdot q} = \frac{\cdot 55}{\sqrt{\beta}}$. Rough and imperfect as this method must be, it may yet serve until further developments of the theory and more perfect experiments, both which are greatly to be desired, shall have substituted certain and logical results.

Values of β.	5	6	8	10	15	20	25	30	40	50	100	200
Increments of S when mass of bar is neglected (p. 486)	·30	·23	·18	·14	·10	·06	·05	·04	·03	·02	·01	·005
Values of $\frac{\cdot 55}{\sqrt{\beta}}$. . .	·25	·22	·19	·17	·14	·12	·11	·10	·09	·08	·05	·04
Total increment of statical deflection.	·55	·45	·37	·31	·24	·18	·16	·14	·12	·10	·06	·045

To apply this Table to any given bridge, the statical deflection due to the greatest load which is liable to pass over it must be ascertained, and also the greatest probable velocity; from these data, and from the length of the bridge, the value of β must be calculated. (See page 483.) The increment of the statical deflection which corresponds to this value of β will be found in the lower line of the above Table.

I will conclude with a few remarks upon the purpose of the preceding pages. The experiments carried out at Portsmouth by Captain James and Lieutenant Galton had given the important and valuable result, that velocity imparted to a load increased the deflections of the bar or bridge over which it passed above those which it would have produced if set at rest upon the same bridge. The amount of this increase was also of so alarming a magnitude, that it seemed incredible that it should have escaped observation in practical cases. Accordingly, when the Commissioners visited the bridges at Ewell and Godstone, the effects there observed, although of the same character, were infinitely less in amount.

It became, therefore, necessary to investigate the laws of these phenomena; and as analysis, even in the hands of so accomplished a mathematician as Mr. Stokes, failed to give tangible results, excepting in cases limited by hypotheses that separated the problem from practical conditions, it became necessary to carry on also experiments directed to the express object of elucidating the theory and tracing its connection with practice. I have already stated that the time which remained to me for this purpose, as well as the limited funds placed in the hands of the Commission, were together insufficient to admit of either constructing the apparatus, or performing the experiments, with the minute and delicate accuracy required for the precise numerical results usually sought for in physical investigations. But my object was rather to elucidate general laws, guided by theory, than to obtain independent numerical results, and I trust that this purpose has been sufficiently answered.

It has been shown that the phenomena in question exhibit themselves in a highly developed state when the apparatus is on a small scale, but that, on the contrary, with the large dimensions of real bridges, their effects are so greatly diminished as to be comparatively of little importance, except in the cases of short and weak bridges traversed with excessive velocities. The theoretical and experimental investigation, which is the subject of the above Essay, will, however imperfect, serve to show that such a diminution of effect, in passing from the small scale to the large, is completely accounted for.

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DRAWINGS DESCRIPTIVE OF THE APPARATUS ERECTED AT PORTSMOUTH DOCKYARD FOR MAKING THE EXPERIMENTS WITH WEIGHTS MOVING WITH DIFFERENT VELOCITIES.

EXPLANATION OF THE PLATES.

The several velocities with which the Car ran over the trial bars C, were obtained by raising it to different heights on the inclined plane A and then releasing it. When the Car had passed the trial bars C, and commenced to ascend the inclined plane B, the moveable rails at D & D were shifted over so as to cause the Car, on its return, to run on the side line till it came to rest.

The deflections of the Bar were taken by brass pencils attached to the Bar at five points, a, b, c, d, e, which traced on metallic paper fixed to the vertical board E, the latter when the Car approached the trial bars, was set in motion by the lever H releasing the weight G.

The velocity of the Car was thus measured: the roller P, struck the lever M which pushed the plate K, from under the pencil L and allowed it to come in contact with, and trace a line upon, the paper on the cylinder O, which revolved equably by clockwork; when the Car had passed, the roller P, by striking the lever N raised the pencil from the cylinder.

FIG. 1. Side Elevation

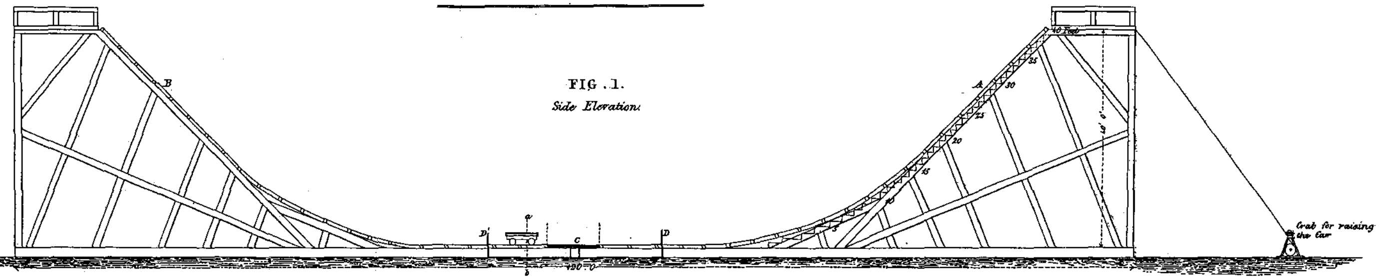


FIG. 2. Plan

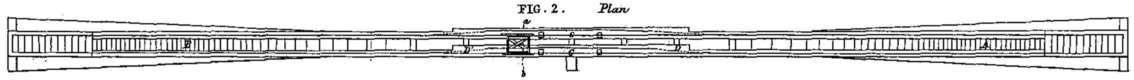
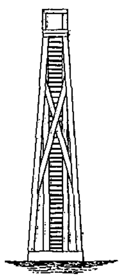
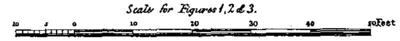


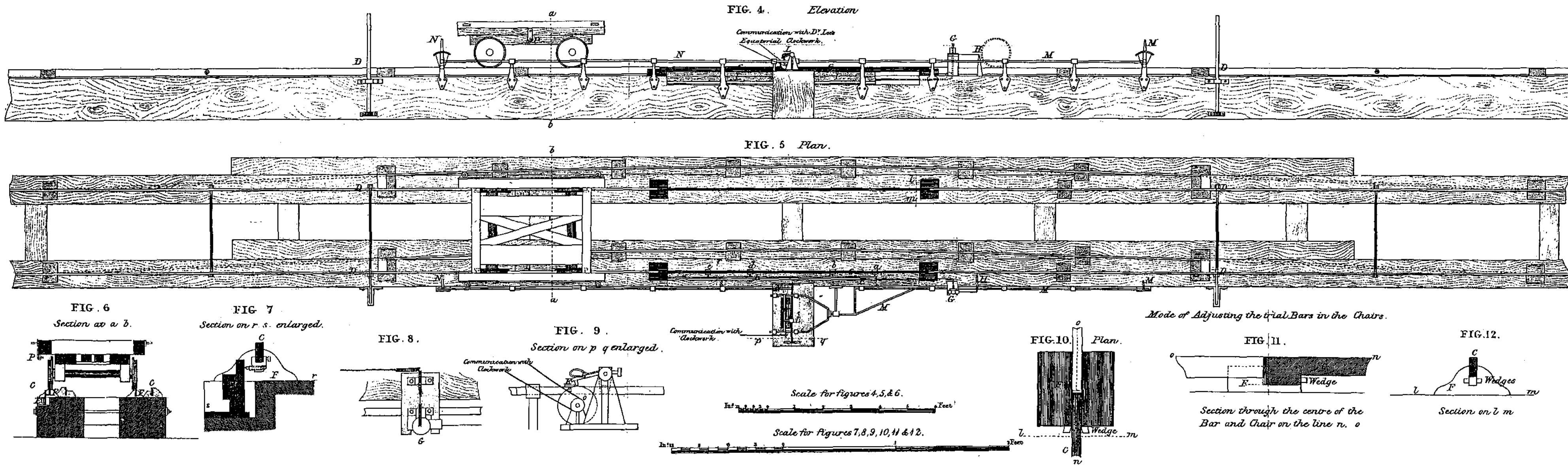
FIG. 3. Rear Elevation

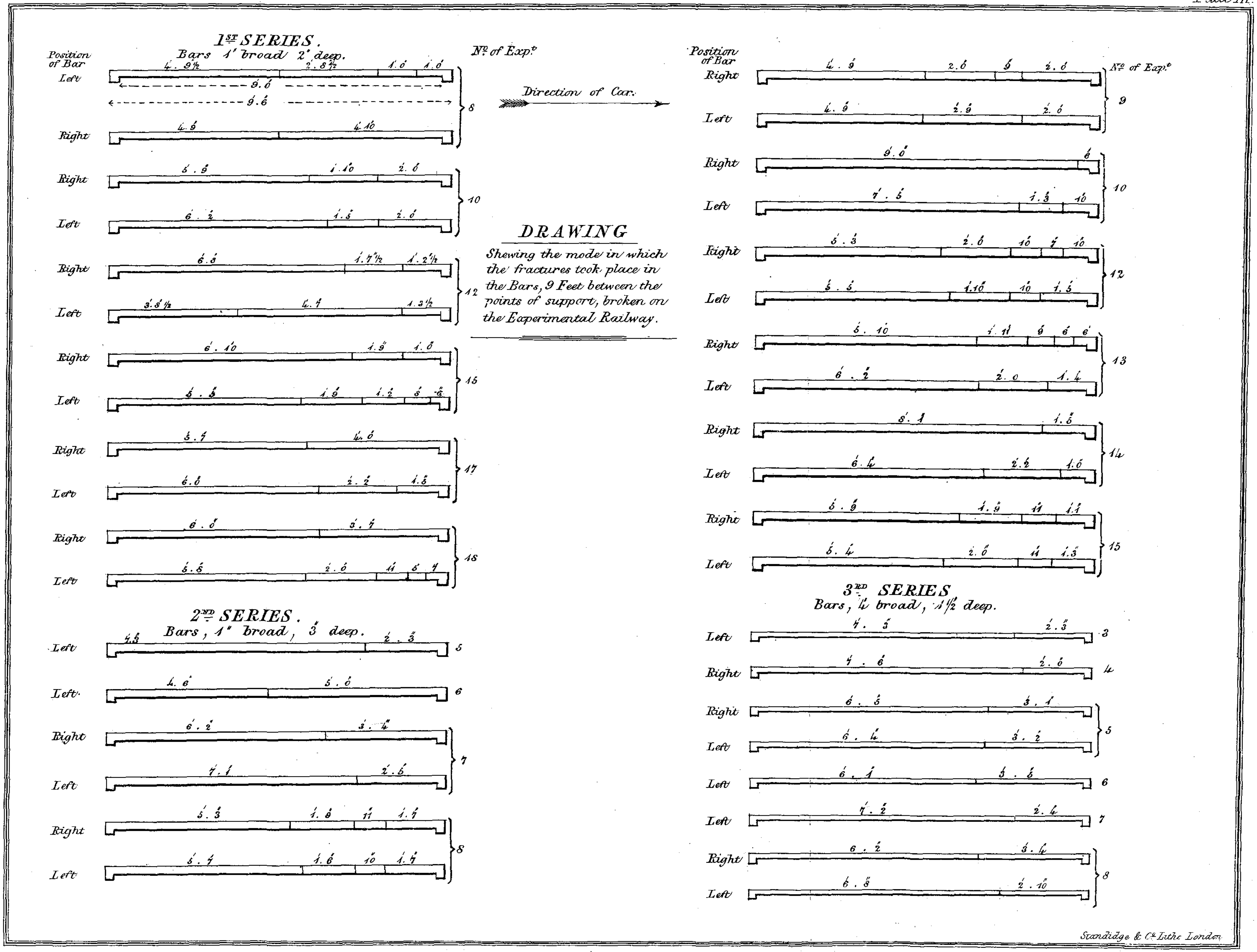


Scale for Figures 1, 2 & 3.



DRAWINGS ON AN ENLARGED SCALE DESCRIPTIVE OF THE ARRANGEMENT FOR REGISTERING THE DEFLECTIONS OF THE TRIAL BARS AND FOR OBTAINING THE VELOCITY OF THE CAR.




1st SERIES.
Bars 1" broad 2" deep.
Position of Bar
No. of Expt.
Direction of Car.
DRAWING
Shewing the mode in which the fractures took place in the Bars, 9 Feet between the points of support, broken on the Experimental Railway.
2nd SERIES.
Bars, 1" broad, 3" deep.
3rd SERIES
Bars, 4" broad, 1½" deep.
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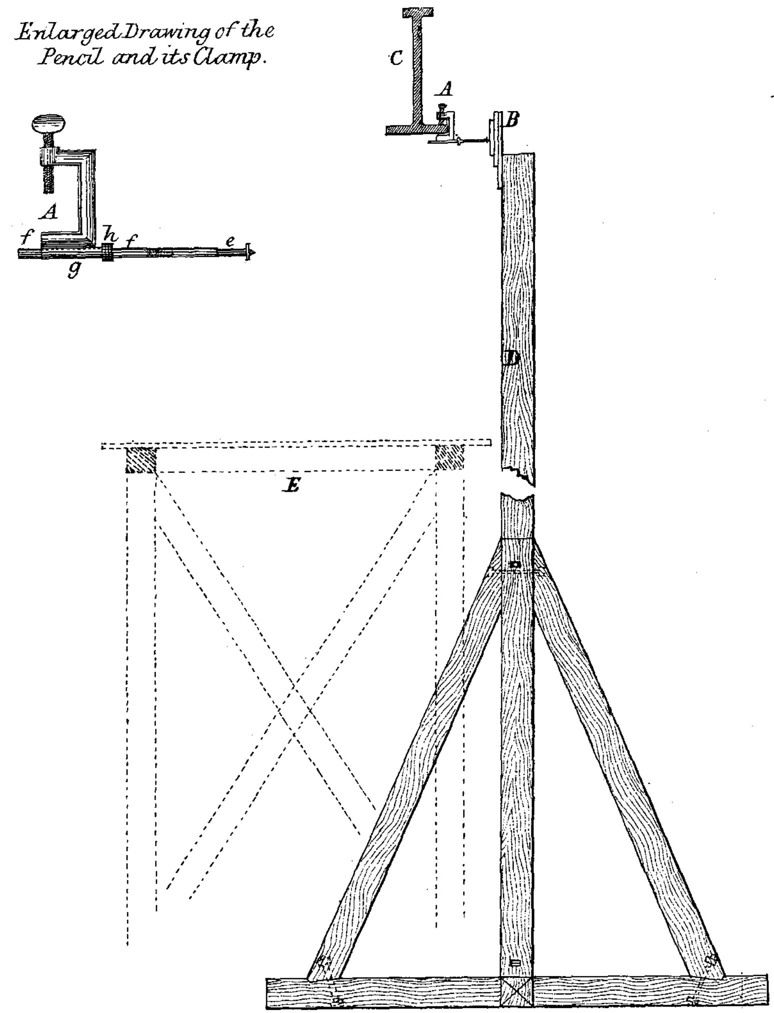
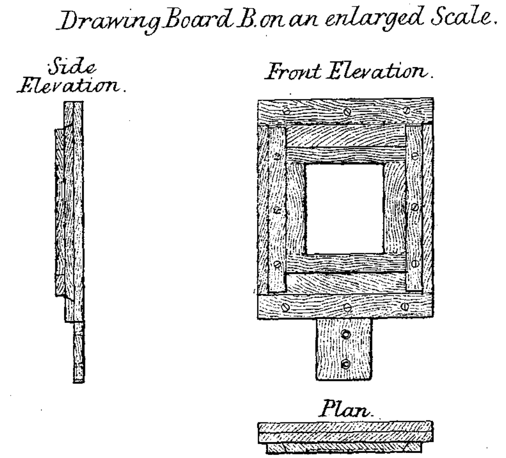
DRAWING OF THE APPARATUS EMPLOYED IN OBSERVING THE DEFLECTIONS OF THE EWELL & GODSTONE BRIDGES.





A. Pencil and Clamp. The Pencil e was fitted very exactly into a case f, in which was a spring which caused the pencil when adjusted in position to press against the paper on the Drawing Board, the adjustment was performed by sliding the case f either backwards or forwards as was required in the cylinder g which was attached to the Clamp and fixing it in the proper position by means of the screw h. The pencil was of brass, and registered on metallic paper prepared by Messrs Harwood.

B. Drawing Board fitted with a Zinc plate for the paper to be against, and adapted to slide horizontally and vertically for purposes of adjustment.

C. Section of Girder.

D. Support for Drawing Board.

E. Stage for Observer.

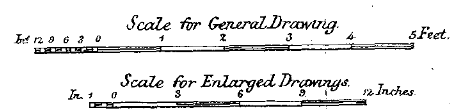


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FIG. 1.

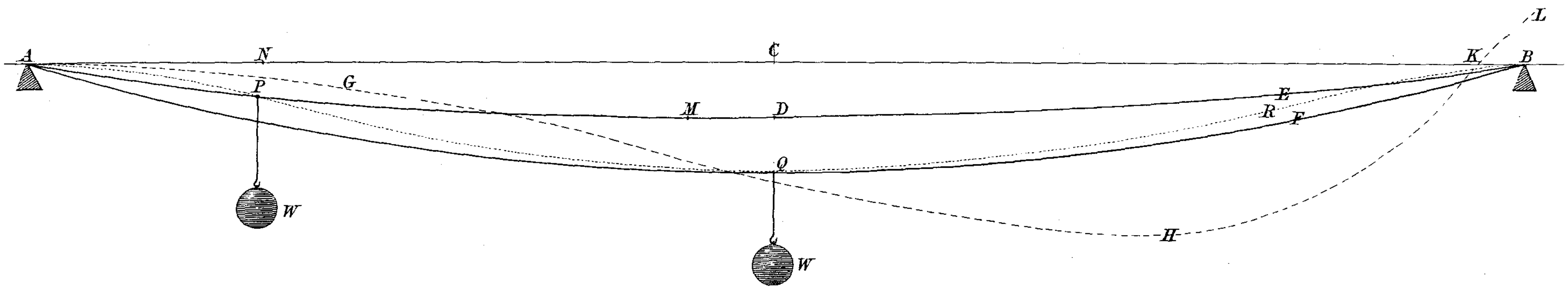


FIG 2.

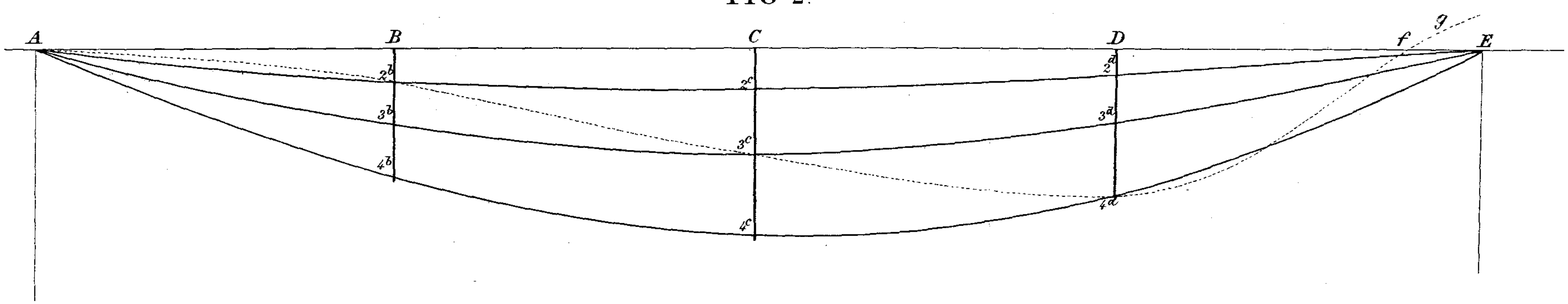
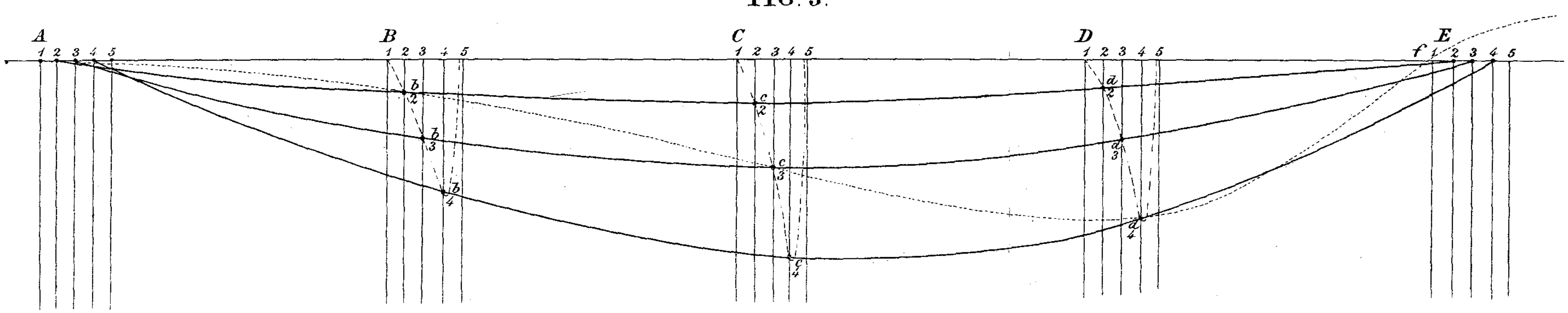


FIG. 3.



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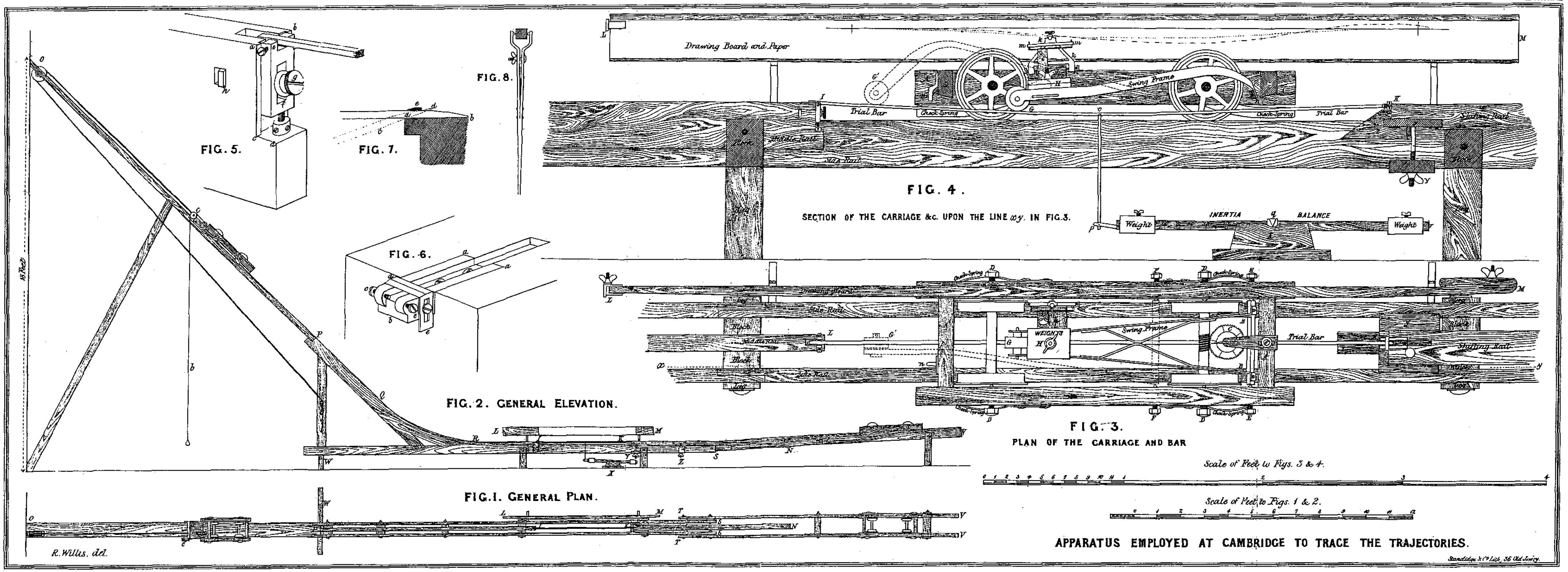

APPARATUS EMPLOYED AT CAMBRIDGE TO TRACE THE TRAJECTORIES.

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Plate VII

Fig. 4. Curves corresponding to the nine Values of B in Mr. Stokes' Tables V & VI.

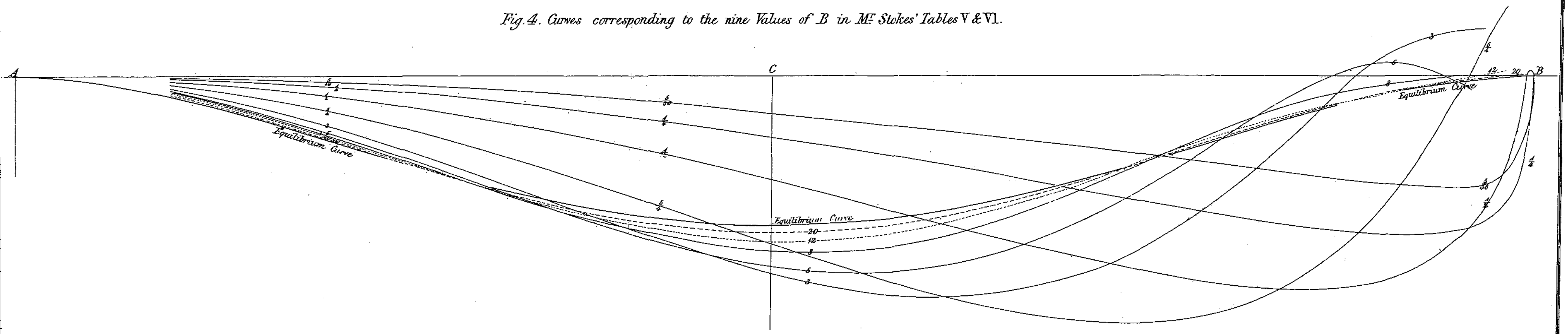
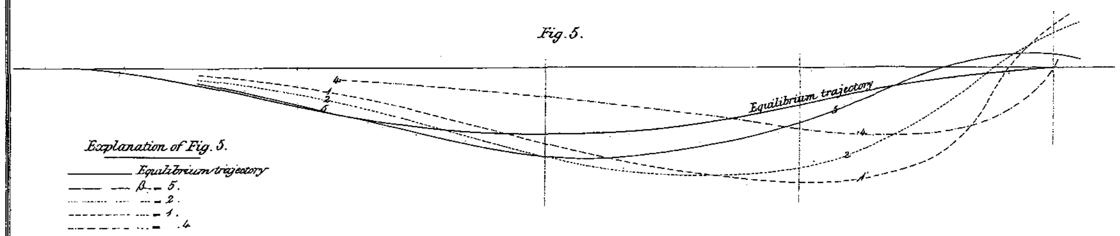


Fig. 5.



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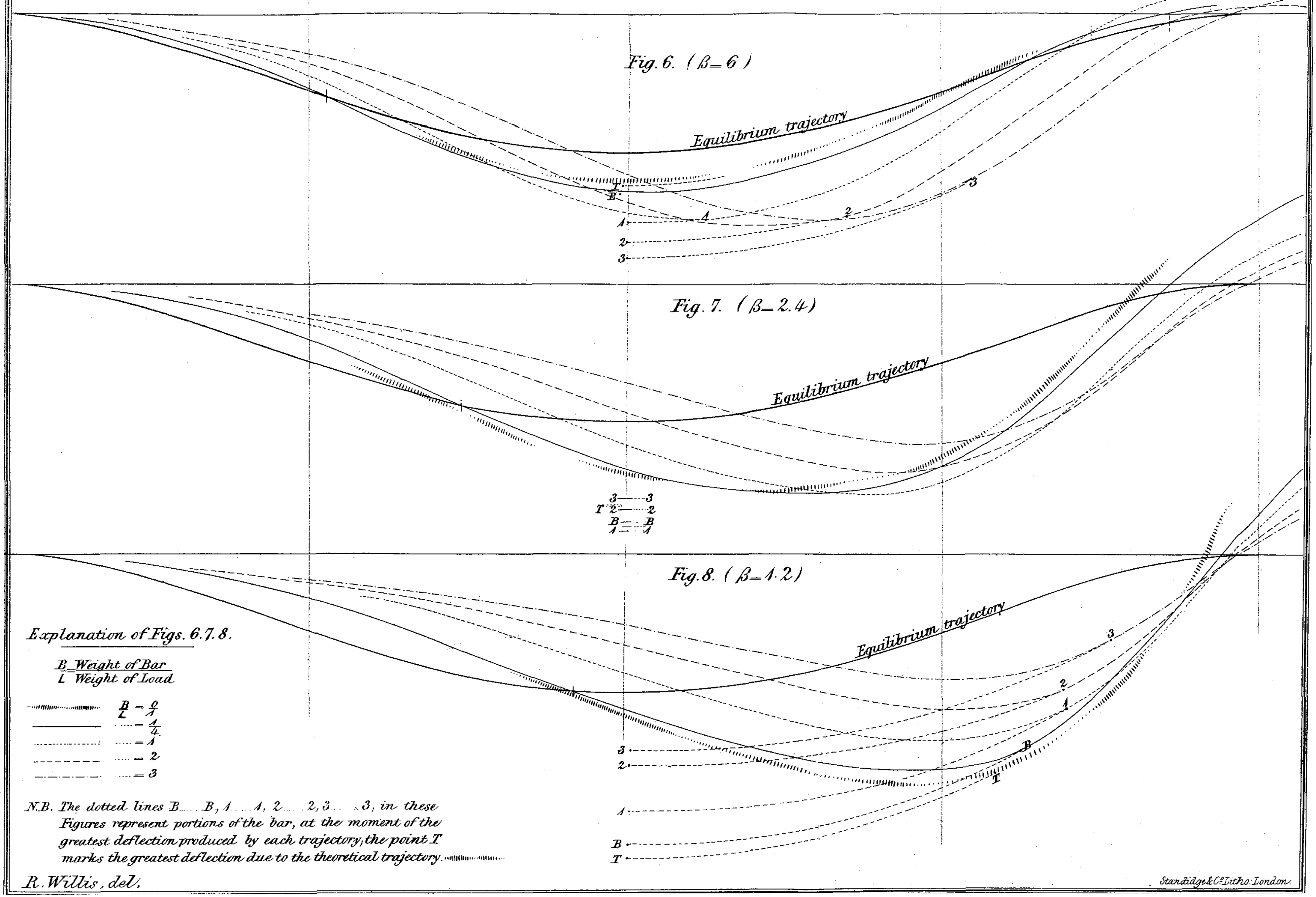
TRAJECTORIES OBTAINED BY THE APPARATUS IN PLATE VI.



CURVES CORRESPONDING TO Mr. STOKES' TABLES A & B.

Fig. B.

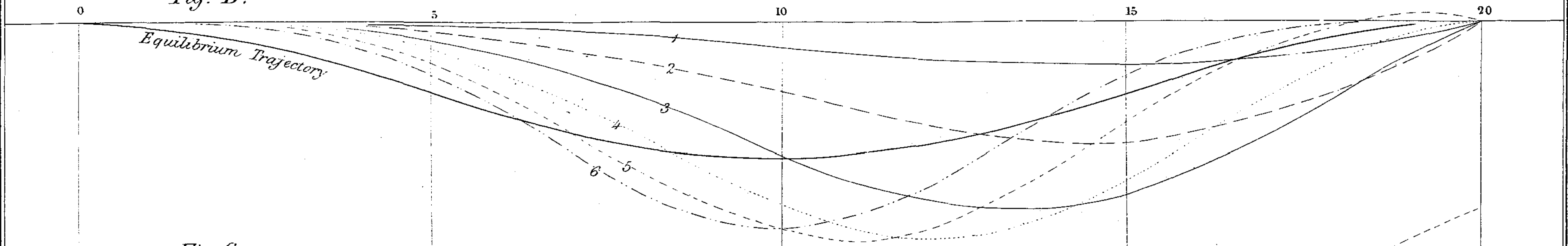


Fig. C.

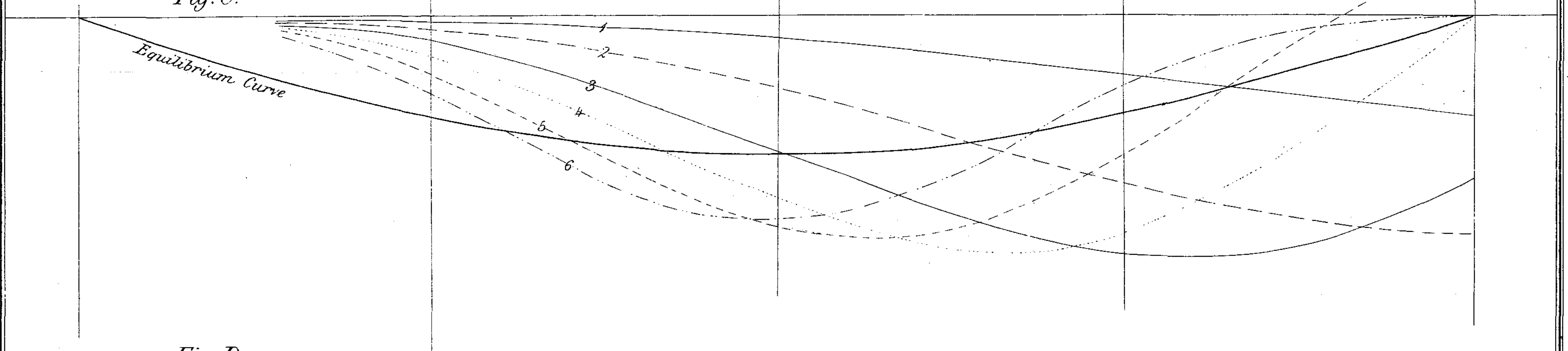


Fig. D.

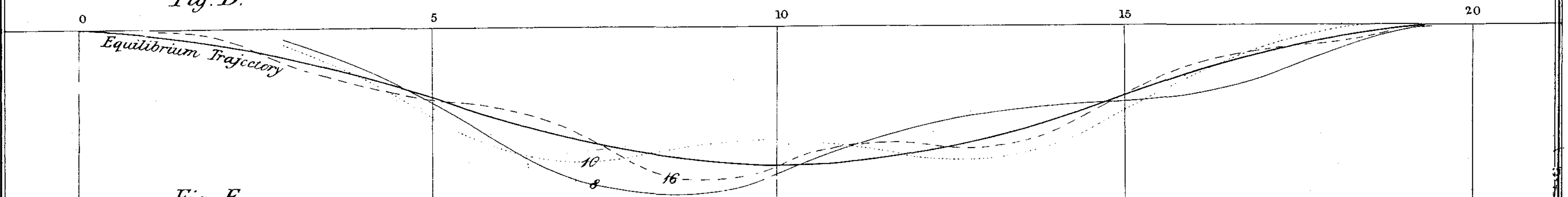
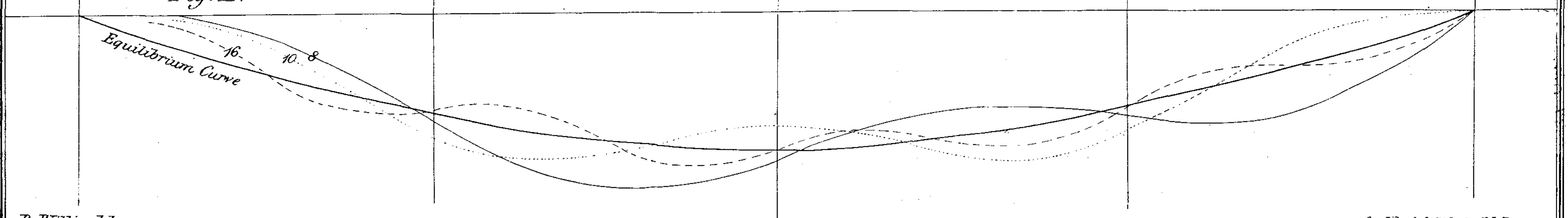


Fig. E.



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